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Behavior of Cylindrical Shells Under Dynamic Loading by Hydrostatic Pressure or by Axial Compression

V. L. AGAMIROV and
A. S. VOL'MIR

A closed circular cylindrical shell, subject to hydrostatic pressure, is considered. It is assumed that the intensity of the pressure increases rapidly with time; thus the shell is subject to dynamic loading. In an article by Lavrent'ev and Ishlinskii (1)[1] dating back to 1949, it was shown that when a load is suddenly applied to a rod or cylindrical shell, the form of loss of stability should be different than in static loading. Later on Hoff published several papers (2, 3), in which he investigated the stability of rods under dynamic loading by a compressing force; it was assumed that the rod had a certain initial deflection. An article by Chawla (4) was devoted to the same problem. In a paper by one of the authors (5) the stability of a cylindrical panel was investigated under the action of a dynamic axial force, and nonlinear shell theory formalism was used. A series of problems on the stability of cylindrical panels under dynamic loading was solved by Bolotin and others (6). The present paper also deals with the nonlinear formulation of the behavior of a closed shell under a rapidly increasing hydrostatic pressure. The inertia of the shell elements as they move in a radial direction is taken into account. At the same time, the inertia terms corresponding to the displacements of points of the shell along the generatrix and along the arc are neglected; this is in agreement with the general concepts of the theory of shells with large deflection. Analogous results were obtained for the case where the shell is subject to dynamic application of an axial compression load.

1 Initial Relations

LET A circular cylindrical shell have an initial deflection comparable with the thickness of the shell. The ends of the shell are hinged to frames; we assume that as the shell is deformed the end sections remain circular, and that the points of these sections can undergo a certain radial displacement. The equation of motion of the shell element is taken in the following form (5)

$$D\nabla^2\nabla^2(w_1 - w_0) = h\left(\frac{\partial^2 w_1}{\partial x^2}\frac{\partial^2 \Phi}{\partial y^2} + \frac{\partial^2 w_1}{\partial y^2}\frac{\partial^2 \Phi}{\partial x^2} - 2\frac{\partial^2 w_1}{\partial x \partial y}\frac{\partial^2 \Phi}{\partial x \partial y} + \frac{1}{R}\frac{\partial^2 \Phi}{\partial x^2} + \frac{q}{h} - \frac{\gamma}{g}\frac{\partial^2 w_1}{\partial t^2}\right) \quad [1.1]$$

and the equation of compatibility of deformation is taken in the form

$$\frac{1}{E}\nabla^2\nabla^2\Phi = \left[\left(\frac{\partial^2 w_1}{\partial x \partial y}\right)^2 - \frac{\partial^2 w_1}{\partial x^2}\frac{\partial^2 w_1}{\partial y^2}\right] - \left[\left(\frac{\partial^2 w_0}{\partial x \partial y}\right)^2 - \frac{\partial^2 w_0}{\partial x^2}\frac{\partial^2 w_0}{\partial y^2}\right] - \frac{1}{R}\left(\frac{\partial^2 w_1}{\partial x^2} - \frac{\partial^2 w_0}{\partial x^2}\right) \quad [1.2]$$

Here

t	=	time
x and y	=	coordinates reckoned along generatrix and along arc, respectively
$w_1(x, y, t)$	=	total deflection of points of midsection
$w_0(x, y)$	=	initial deflection
Φ	=	stress function in midplane
D	=	flexural rigidity
R and h	=	radius of midplane and thickness of shell, respectively
γ	=	weight per unit volume of material
$q(t)$	=	intensity of external uniformly distributed pressure
∇^2	=	two-dimensional Laplacian operator

We assume that the initial deflections are distributed in accordance with the following law.

$$w_0 = f_0(\sin \alpha x \sin \beta y + \psi \sin^2 \alpha x + \varphi) \quad [1.3]$$

where

α	=	π/L
β	=	n/R
n	=	number of waves along periphery
L	=	length of shell

The initial deflection f_0 is assumed specified. For the total deflection we assume an analogous relation

$$w_1 = f_1(\sin \alpha x \sin \beta y + \psi \sin^2 \alpha x + \varphi) \quad [1.4]$$

The total deflection f_1 is considered a function of time; the parameters ψ and φ in expressions [1.3 and 1.4] are identical. It is thus assumed that the form of the initial deflection is not fixed; it is assumed to be "in resonance," so to speak, with the form of the bent surface for each instant of time (7). An approximating expression for a deflection of type [1.4] has been used in several earlier papers on the statics of shell stability under external pressure (7–9); the first term in the parentheses corresponds to the solution of the stability problem "in the small," the second reflects the preferred inward bulging of the shell when stability is lost "in the large," and the third term takes into account the radial displacements of the points of the end sections. The hinged joint condition is not completely satisfied, owing to the presence of the second term; however, as shown by investigations of the statical

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[1] Numbers in parentheses indicate References at end of paper.

problem (10), this circumstance should not influence greatly the results of the solution.

2 Determination of Stress Function. Continuity Condition

We substitute expressions [1.3 and 1.4] into the right half of Equation [1.2]. Then the integral of this equation can be represented in the following form (10)

$$\Phi = E(K_1 \cos 2\,\alpha x + K_2 \cos 2\,\beta y + K_3 \sin \alpha x \sin \beta y + K_4 \sin 3\,\alpha x \sin \beta y) - \frac{qR}{2h}\,x^2 - \frac{qR}{4h}\,y^2 \quad [2.1]$$

Here

$$K_1 = \frac{1}{16\alpha^2}\left[\frac{(f_1 - f_0)^2\beta^2}{2} + (f_1 - f_0)f_0\beta^2 - \frac{2(f_1 - f_0)\psi}{R}\right]$$

$$K_2 = \frac{\alpha^2}{32\beta^2}\,(f_1^2 - f_0^2) \qquad K_4 = \frac{\alpha^2\beta^2}{(9\alpha^2 + \beta^2)^2}\,(f_1^2 - f_0^2)\psi$$

$$K_3 = \frac{\alpha^2(f_1 - f_0)}{R(\alpha^2 + \beta^2)^2} - \frac{\alpha^2\beta^2(f_1^2 - f_0^2)}{(\alpha^2 + \beta^2)^2}\,\psi$$

..... [2.2]

The last two terms in [2.1] correspond to the stresses in the middle surface, determined by membrane theory.

Let us write down the expression for the strain ϵ_y in the midsurface

$$\epsilon_y = \frac{\partial v}{\partial y} + \frac{1}{2}\left(\frac{\partial w_1}{\partial y}\right)^2 - \frac{1}{2}\left(\frac{\partial w_0}{\partial y}\right)^2 - \frac{w}{R} \quad [2.3]$$

Here v is the displacement of the points of the middle surface along the arc and $w = w_1 - w_0$. On the other hand, within the elastic limit, we have

$$\epsilon_y = \frac{1}{E}\left(\frac{\partial^2\Phi}{\partial x^2} - \mu\,\frac{\partial^2\Phi}{\partial y^2}\right) \quad [2.4]$$

where μ is the Poisson coefficient. Hence

$$\frac{\partial v}{\partial y} = \frac{1}{E}\left(\frac{\partial^2\Phi}{\partial x^2} - \mu\,\frac{\partial^2\Phi}{\partial y^2}\right) - \frac{1}{2}\left(\frac{\partial w_1}{\partial y}\right)^2 + \frac{1}{2}\left(\frac{\partial w_0}{\partial y}\right)^2 + \frac{w}{R} \quad [2.5]$$

The continuity condition, pertaining to the displacement v, has the following form

$$\oint \frac{\partial v}{\partial y}\,dy = 0 \quad [2.6]$$

Inserting Equation [2.5] and taking into account the preceding relations, we get

$$\varphi = \frac{R^2}{h(f_1 - f_0)}\,\frac{q}{E}\left(1 - \frac{\mu}{2}\right) - \frac{\psi}{2} + \frac{R}{8}\,(f_1 + f_0)\beta^2 \quad [2.7]$$

Thus, the parameter φ is expressed in terms of ψ and f_1. It remains to formulate equations for these parameters.

3 Use of the Bubnov-Galerkin Method

An approximate integration of Equation [1.1] can be performed by use of the Bubnov-Galerkin method, using the equations

$$\int_0^L \int_0^{2\pi R} X \sin \alpha x \sin \beta y\,dx\,dy = 0 \quad [3.1]$$

$$\int_0^L \int_0^{2\pi R} X \sin^2 \alpha x\,dx\,dy = 0 \quad [3.2]$$

where X denotes

$$X = D\nabla^2\nabla^2(w_1 - w_0) - h\left(\frac{\partial^2 w_1}{\partial x^2}\,\frac{\partial^2\Phi}{\partial y^2} + \frac{\partial^2 w_1}{\partial y^2}\,\frac{\partial^2\Phi}{\partial x^2} - 2\,\frac{\partial^2 w_1}{\partial x \partial y}\,\frac{\partial^2\Phi}{\partial x \partial y} + \frac{1}{R}\,\frac{\partial^2\Phi}{\partial x^2} + \frac{q}{h} - \frac{\gamma}{g}\,\frac{\partial^2 w_1}{\partial t^2}\right) \quad [3.3]$$

We substitute in Equation [3.3] the foregoing expressions for w_1, w_0 and Φ. Carrying out the integration in Equation [3.1], we obtain the following equation that relates the parameters of the deflection with the time varying load

$$q^\circ = C_0\left(1 - \frac{\zeta_0}{\zeta_1}\right) + C_1(\zeta_1^2 - \zeta_0^2) + C_2(\zeta_1^2 - \zeta_0^2)\psi^2 - C_3\,(\zeta_1 - \zeta_0)\psi + C_4\,\frac{1}{\zeta_1}\,\frac{\partial^2\zeta_1}{\partial t^2} \quad [3.4]$$

where

$$C_0 = \frac{1}{1 + \frac{1}{2}\xi^2}\left[\frac{\xi^8}{\pi^2(1 + \xi^2)^2\delta} + \frac{\pi^2}{12(1 - \mu^2)}\,\frac{(1 + \xi^2)^2}{\xi^2}\,\delta\right] \quad [3.5]$$

$$C_1 = \frac{\pi^2}{16(1 + \frac{1}{2}\xi^2)}\,\frac{1 + \xi^4}{\xi^2}\,\delta$$

$$C_2 = \frac{\pi^2\xi^2}{1 + \frac{1}{2}\xi^2}\left[\frac{1}{(1 + \xi^2)^2} + \frac{1}{(1 + 9\xi^2)^2}\right]\delta$$

$$C_3 = \frac{1}{4(1 + \frac{1}{2}\xi)^2}\left[1 + \frac{8\xi^4}{(1 + \xi^2)^2}\right]$$

$$C_4 = \frac{\gamma R^2}{(1 + \frac{1}{2}\xi^2)gE\eta}$$

..... 3.6]

Here we have introduced the dimensionless parameters

$$q^\circ = \frac{q}{E}\left(\frac{R}{h}\right)^2 \qquad \zeta_1 = \frac{f_1}{h} \qquad \zeta_0 = \frac{f_0}{h}$$

$$\xi = \frac{\pi R}{nL} = \frac{\alpha}{\beta} \qquad \eta = n^2\,\frac{h}{R} \qquad \delta = \frac{Rh}{L^2}$$

..... [3.7]

The second Bubnov-Galerkin Equation [3.2] will not be expanded here. To simplify the arguments, we assume for the parameter ψ the expression that follows from the solution of the static problem (10)

$$\psi = \frac{N_3\zeta_1}{2(N_2\zeta_1^2 + N_1)} \quad [3.8]$$

where

$$N_1 = \frac{\pi^4}{6(1 - \mu^2)}\,\delta^2 + \frac{1}{8}$$

$$N_2 = C_2\,\frac{\pi^2}{4}\,\frac{1 + \frac{1}{2}\xi^2}{\xi^2}\,\delta^2$$

$$N_3 = \frac{C_3\eta}{4}\left(1 + \frac{\xi^2}{2}\right)$$

..... [3.9]

This means that at a specified value of ζ_1, the waviness of the shell is assumed to be the same for the case of rapid loading as for slow loading.

We substitute [3.8] in [3.4] and leave out the inertia term. We then obtain for $\zeta_0 = 0$ and $\zeta_1 \to 0$ the parameter of the upper critical pressure, found to be $q_*^\circ = C_0$, or

$$q_*^\circ = \frac{1}{1 + \frac{1}{2}\xi^2}\left[\frac{\xi^4}{(1 + \xi^2)^2\eta} + \frac{(1 + \xi^2)^2}{12(1 - \mu^2)}\,\eta\right] \quad [3.10]$$

This expression corresponds to the well-known Mises formula (11).

4 Relation Between Deflection Parameters and Time

We shall assume that the intensity of the pressure increases as

$$q = ct \qquad [4.1]$$

and determines the character of the buckling of the shell as a function of c, the uniform rate of loading in atmospheres per second. We introduce the dimensionless time parameter

$$t^\circ = \frac{ctR^2}{Eh^2 q_*{}^\circ} = \frac{q^\circ}{q_*{}^\circ} = \frac{q}{q_*} \qquad [4.2]$$

and then Equation [3.4] can be rewritten

$$\frac{d^2\zeta_1}{dt^{\circ 2}} - S\left[\left(t^\circ - \frac{\zeta_1 - \zeta_0}{\zeta_1}\right)\zeta_1 - B_1(\zeta_1{}^3 - \zeta_1\zeta_0{}^2) - B_2\frac{\zeta_1{}^5 - \zeta_1{}^3\zeta_0{}^2}{(N_2\zeta_1{}^2 + N_1)^2} + B_3\frac{\zeta_1{}^3 - \zeta_1{}^2\zeta_0}{N_2\zeta_1{}^2 + N_1}\right] = 0 \qquad [4.3]$$

Here

$$B_1 = C_1\frac{\xi^2}{\pi^2}\frac{\eta}{\delta q_*{}^\circ} \qquad B_2 = C_2\frac{\delta}{q_*{}^\circ}\frac{N_3{}^2}{4} \qquad B_3 = \frac{C_2 N_3}{2q_*{}^\circ}$$

$$S = \left(1 + \frac{\zeta^2}{2}\right)q_*{}^{03}\left(\frac{VE}{cR}\right)^2\left(\frac{h}{R}\right)^4\eta \qquad \left(V = \sqrt{\frac{Eg}{\gamma}}\right)$$

. . . . . [4.4]

By V is meant the rate of propagation of the elastic waves in the shell material.

We thus obtain an ordinary nonlinear differential equation, [4.3], for the dimensionless deflection ζ_1. The integration of this equation was carried out by the Adams method under the following initial conditions

$$\zeta_1 = \zeta_0 \qquad \frac{d\zeta_1}{dt^\circ} = 0 \text{ for } t^\circ = 0 \qquad [4.5]$$

It was assumed in the calculation that for aluminum shells $V = 5 \times 10^5$ cm/sec, $\mu = 0.3$ and $E = 7.75 \times 10^5$ kg/cm^2. In addition, we assumed $R/L = 0.45$, $R/h = 112$, $\delta = 1.8 \times 10^{-3}$ and $\zeta_0 = 0.001$.

5 Results of the Calculations

Figs. 1 and 2 show curves of ζ_1 vs. t°, which represent the connection between the total deflection and the time of loading. The first group of the curves in Fig. 1 corresponds to the solution of the static problem as $c \to 0$; this solution can be obtained from Equation [4.3] by setting the inertia term equal to zero. Each curve corresponds to a certain number n of waves formed along the periphery as the shell buckles. We see that the smallest upper critical pressures takes place at $n = 6$ (point A in Fig. 1); it is determined by the parameter $q_*{}^\circ = 0.0402$. The lower critical pressure (point B) has a minimum when $n = 5$. It must be specified here that the curves of Fig. 1 pertain to a shell with initial deflection; however the selected initial deflection amounts to 0.001 of the thickness of the shell, so that the curve $\zeta_1(t^\circ)$ is close to the corresponding curve for an ideally smooth shell.

The same Fig. 1 shows curves for different values of n, determined for the case of dynamic loading at a pressure buildup rate $c = 2 \times 10^3$ atm/sec. All the curves on the section from $t^\circ = 0$ to $t^\circ = 1.5$ merge with the abscissa axis at the chosen value of ζ_0. We recall that according to Equation [4.2] the quantity t° defines the ratio of the current pressure q to the

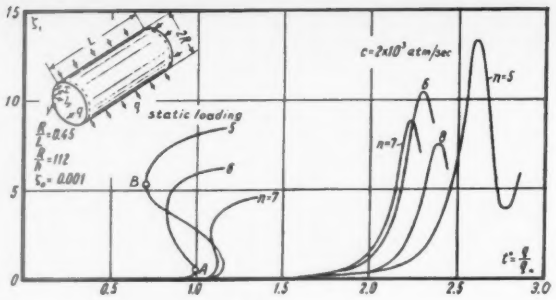

Fig. 1

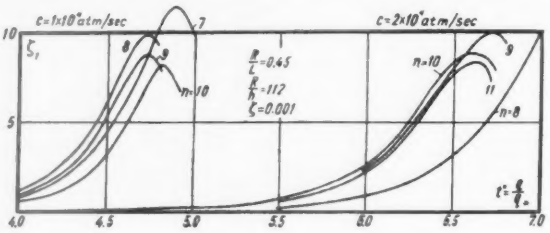

Fig. 2

upper critical pressure q_*, calculated for $n = 6$. As seen from Fig. 1, the first to deviate from the abscissa axis is the curve $n = 7$. Corresponding to this curve are the lowest values of q for all values of ζ_1, up to $\zeta_1 = 9$. One can assume that the buckling of real shells at a given rate of loading and for a given initial deflection will be accompanied by the formation of seven waves on the periphery, instead of six or five waves, which are produced in static loading. The vigorous increase in the deflections takes place at a parameter t° ranging from 2.0 to 2.2. The critical pressure is thus approximately twice the upper critical load. After reaching the maximum deflection, corresponding to a given number of waves, the shell begins to execute nonlinear oscillations. This is shown in Fig. 1 for the case $n = 5$. However, we shall identify the primary process of sharp increase in deflection with the loss of shell stability and shall not be interested in the subsequent oscillatory process. We note that the buckling of metallic shells will be accompanied as a rule by the development of plastic deformations, the consideration of which is outside the scope of this discussion.

Fig. 2 shows two other series of $\zeta_1(t^\circ)$ curves, pertaining to higher loading rates, $c = 10^4$ and 2×10^4 atm/sec. The outermost curves on the left are those for $n = 8$ in the former case and $n = 10$ in the latter. The zone of rapid rise in deflections occurs respectively at $t^\circ = 4.5$ to 4.7 and at $t^\circ = 6.2$ to 6.6. Consequently, the dynamic effect manifests itself in a successive increase in the number of waves, corresponding to higher forms of stability loss, and in a considerable increase in the critical pressure (in the latter case by a factor of more than six times).

V. E. Mineev performed experiments on the dynamic loading of samples of the same dimensions which were chosen in the foregoing for the calculations. The theoretical results obtained were confirmed not only qualitatively, but in part also quantitatively.

6 Case of Axial Compression

Analogous results were obtained when solving the problem of the stability of the shell in dynamic axial loading. In

Equation [1.1] we put $q = 0$. The expressions for the initial and total deflection are taken in the form

$$w_0 = f_0 (\sin \alpha x \sin \beta y + \psi \sin^2 \alpha x \sin^2 \beta y)$$

$$w_1 = f_1 (\sin \alpha x \sin \beta y + \psi \sin^2 \alpha x \sin^2 \beta y) \quad \ldots\ldots [6.1]$$

where unlike Equation [1.3], $\alpha = m\pi/L$, where m is the number of half-waves in the axial direction.

We add an additional term $(-py^2/2)$ to Equation [2.1] for Φ; p is the average axial compressive stress.

We assume that p varies with time as $p = kt$. We use the dimensionless parameters

$$t^\circ = \frac{ktR}{Ehp_*{}^\circ} \qquad p^\circ = \frac{p}{E}\frac{R}{h} \qquad \zeta_0 = \frac{t_0}{h}$$

$$\zeta_1 = \frac{f_1}{h} \qquad \eta = n^2\frac{h}{R} \qquad \xi = \frac{m\pi R}{nL} = \frac{\alpha}{\beta} \quad \ldots\ldots [6.2]$$

where $p_*{}^\circ$ denotes the upper critical stress.

Equation [4.3] for the deformation now becomes

$$\frac{d^2\zeta_1}{dt^{\circ 2}} - S_1\left[t^\circ\zeta_1 - (\zeta_1 - \zeta_0) - \frac{A_2}{A_1}(\zeta_1{}^3 - \zeta_1\zeta_0{}^2) - \frac{A_3}{A_1}\psi^2\left(\zeta_1{}^3 - 27\frac{\zeta_0{}^4}{\zeta_1} + 27\,\zeta_0{}^3 - 10\zeta_1\zeta_0{}^2 + 9\frac{\zeta_0{}^5}{\zeta_1{}^2}\right) - \frac{A_4}{A_1}\psi(\zeta_1{}^2 - \zeta_0{}^2)\right] = 0 \quad [6.3]$$

Here

$$A_1 = \frac{(1+\xi^2)^2\eta}{12(1-\mu^2)\xi^2} + \frac{\xi^2}{\eta(1+\xi^2)^2} \qquad A_2 = \frac{1+\xi^4}{16\xi^2}\eta$$

$$A_3 = \frac{\eta}{16\xi^2}\left[1 + \xi^4 + \frac{16\xi^4}{(1+\xi^2)^2} + \frac{36\xi^4}{(1+9\xi^2)^2} + \frac{36\xi^4}{(9+\xi^2)^2}\right]$$

$$A_4 = -\frac{1}{8\xi^2}\left[1 + \frac{16\xi^4}{(1+\xi^2)^2}\right]$$

$$S_1 \simeq A_1{}^3\eta\xi^2\left(\frac{h}{R}\right)^2\left(\frac{V}{k}\frac{E}{R}\right)^2 \quad \ldots\ldots [6.4]$$

The value of ψ is determined either from an equation similar to [3.2] or from the solution of the static problem (10). The derivations were made for various values of ξ and η. The character of the ξ_1 vs. t° curve, obtained by integrating Equation [6.3], is the same as in the preceding problem. The case was also considered in which the time variation of the relative approach of the ends of the shell is specified.

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Reviewer's Comment

The paper presents relatively simple approximate solutions for problems of initially deformed, thin walled, circular cylindrical shells under dynamic loading either by external hydrostatic pressure or by axial compression. The analyses are performed for loads which increase linearly with time. All of the computed results presented are concerned with the hydrostatic pressure loading.

In this country related problems involving static loading have been analyzed for the most part with the aid of the Rayleigh-Ritz procedure [see, for example, (1–7)], while in the Soviet Union the Bubnov-Galerkin method appears to have more widespread appeal. Agamirov and Vol'mir apply the latter method.

The basic relations used in their work are similar to those of (1–7). However, the degrees of freedom included in the analysis, with regard to both the assumed deflection function and the stress function, are more restricted. By assuming that the amplitude of the nonuniform component of the axisymmetric terms of the assumed deflected shape remains related to the amplitude of the asymmetric component in a manner identical to that determined for the related statical loading, and by selecting the complementary part of the solution of the stress function on the basis of elementary membrane theory, the authors reduce the problem to one involving only a single degree of freedom. Furthermore, as pointed out in the text, the deflection function does not satisfy hinged joint conditions, but rather conditions lying somewhere between those for free and clamped ends. These assumptions can restrict the quantitative and, possibly, the qualitative results significantly. It is also unfortunate that the function representing the initial deformation is taken to be identical to that for the total radial deflection except for one parameter (compare Eqs. [1.3 and 1.4]). An investigation of the interesting case corresponding to the consideration of purely axisymmetric initial deformation is thus precluded.

In spite of the shortcomings of the analyses presented, the results are of interest. They show that under conditions of high dynamic loading rates, the loads at which deformations first become pronounced can be significantly larger than the corresponding loads determined with static loading. It is unfortunate that the experiments, which are stated to be in agreement with the theory, are not described. It is likewise regrettable that related dynamic analyses which are known to the reviewer appear only in the classified literature.

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Nonlinear Differential Equations of the Automatic Regulation System Having a Single Regulating Member

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THIS WORK provides a rigorous mathematical justification of the Lurye method (1).[1] In addition, unlike (1), the article includes the following:

1 An exposition, in an invariant form, without selection of any special coordinate system. This made it possible to write the system of resolving equations directly from the coefficients of the given system of differential equations, regardless of the canonical character of the matrix for coefficients of the open loop system.

2 A more detailed study of the system for resolving quadratic equations showing that stability on the whole is sufficiently specified if a certain algebraic equation has two real roots.

3 A more detailed study of the zero roots for the characteristic equation of the open loop system.

4 An extension of Lurye's method which, with some complications in computations, leads to sufficient conditions for stability. These include all stability conditions which can be obtained by means of a definite form of the Lyapunov function ("the quadratic form of the basic coordinates plus an integral due to nonlinearity").[2]

Algebraic calculations, connected with obtaining concrete conditions for coefficients, grow rapidly with increases in the order of the system and, *in general*, for systems of an order of ≥ 5 it is extremely cumbersome. Nevertheless, these conditions can always be obtained by the finite number of rational algebraic operations.

For systems of a larger order one can hardly hope to obtain stability conditions, in general, in the form of inequalities on coefficients because their derivation and final form are extremely awkward. This is explained, to a large extent, by the large amount of undetermined parameters.

Evidently, it is more desirable to have a method of obtaining sufficiently "broad" similar conditions in the form of sequences of determined rules. This method makes it possible, in each concrete case when the amount of undetermined parameters is small, to obtain stability conditions in the form of certain inequalities, connecting these parameters.

In our treatment, Lurye's method is, in fact, of this type.

The author considers it his duty to emphasize that, in basic outline, his work follows Lurye's work (1).

Following completion of this article the author became acquainted with the voluminous and interesting monographs of Letov (2) and Bromberg (3) where we can find a series of common features with this article. The basic results of this article are briefly explained in (4 and 5).

Section 1 Formulation of the Problem

1′ Let us agree upon the following notations. Where no reservations are made to the contrary we will henceforth designate matrices by large Latin letters, vectors by small letters, and scalar quantities by Greek letters. Exceptions will be

t	=	time
n	=	order of vectors and matrices
m, i, j	=	for integers, indexes of summation
V	=	Lyapunov function
$dV/dt = -W$	=	its derivative

All matrices, vectors and numbers are assumed to be real; A^* denotes a transposed matrix. Sometimes, complex matrices will also be considered and then A^* will denote a transposed and complex-conjugate matrix. Vector components will be arranged in the form of columns (denoted by a) or in the form of rows (a^*). Then $a^*b = (a, b)$ is a scalar product and ab^*, the matrix (unit rank).

2′ Following Lurye, we will study the systems of equations of "indirect automatic regulation."[3]

$$dy/dt = Ay + a\xi$$
$$d\xi/dt = \varphi(\sigma) \qquad \sigma = b^*y - \rho\xi \qquad [1.1]$$

Here $\varphi(\sigma)$ is a continuous function of σ, $-\infty < \sigma < \infty$. Later, additional restrictions will be imposed on the function

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[1] Numbers in parentheses indicate References at end of paper.

[2] Some of the statements (1), while correct, require more detailed proof.

[3] See (1), p. 60, system [1.1].

$\varphi(\sigma)$.[4] The feedback coefficient $\rho > 0$.

By substituting $x = dy/dt$ the system [1.1] comes to the nonlinear system of $n + 1$ equations

$$\begin{aligned} dx/dt &= Ax + a\varphi(\sigma) \\ d\sigma/dt &= b^*x - \rho\varphi(\sigma) \end{aligned} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots [1.2]$$

We will assume that the matrix

$$\begin{pmatrix} A & a \\ b^* & -\rho \end{pmatrix} \qquad [1.3]$$

has an inverse.[5] Because x, σ are connected with y, ξ by relationships

$$\begin{aligned} x &= Ay + a\xi \\ \sigma &= b^*y - \rho\xi \end{aligned}$$

the systems [1.1 and 1.2] are fully equivalent.

Henceforth, we will be concerned only with systems in [1.2].

3′ Following Lurye, we will solve the following problem. *It is necessary to determine the conditions, imposed on coefficients of the system in* [1.2], *so that for any continuous function* $\varphi(\sigma)$, *subject to conditions*

$$\varphi(0) = 0 \qquad \sigma\varphi(\sigma) > 0 \qquad \text{when } \sigma \neq 0 \qquad [1.4]$$

the trivial solution of the system [1.2] *be "on the whole stable," i.e., so that the trivial solution* $x \equiv 0$, $\sigma \equiv 0$ *be stable according to Lyapunov. Also any solution must be continuous when* $t \to \infty$ *and it must tend to zero, so that* $x(t) \to 0$, $\sigma(t) \to 0$ *when* $t \to \infty$.

Assuming in [1.2] $\varphi(\sigma) = \mu\sigma$, we obtained a linear system with a matrix

$$A_{\mu}{}^{(0)} = \begin{pmatrix} A & a\mu \\ b^* & -\rho\mu \end{pmatrix} \qquad [1.5]$$

In order to solve the formulated problem it is evidently necessary that for any μ, $0 < \mu < \infty$, all the characteristic values of matrix [1.5] have negative real parts. Hence, it follows that matrix [1.3], in particular, has an inverse.

Let us designate $\lambda_1, \ldots, \lambda_n$ by means of the characteristic values of matrix A. In order to solve the formulated problem it is necessary that

$$\mathrm{Re}\lambda_j \leq 0 \qquad j = 1, \ldots, n \qquad [1.6]$$

Actually, in the reverse situation where there is a sufficiently small $\mu > 0$ some of the characteristic values of matrix [1.5] would have positive real parts.

4′ However, cases can be encountered where some of the $\mathrm{Re}\lambda_i > 0$. This means that in the absence of regulation the system is unstable and there is no stability even at the boundary. Let us suppose, however, that for some mean values of servomotor time constant the operation of a closed linear system is stable, i.e., that when some μ are greater than 0, matrix [1.5] has all characteristic values with negative real parts. On the μ axis there would then be one or several zones of stability; this is determined by conventional methods (6, 7).

Suppose that we have one zone of stability[6] of the form (μ_0, ∞). This means that when $\mu_0 < \mu < \infty$, matrix [1.5] has characteristic values with negative real parts and when $\mu = \mu_0$ some of the proper values are frequently imaginary or zero. Then, in formulating analogous problems, instead of the conditions in [1.4] we would have

$$\begin{aligned} \varphi(0) &= 0 \qquad \varphi(\sigma) > \mu_0\sigma \qquad \text{when } \sigma > 0 \\ \varphi(\sigma) &< \mu_0\sigma \qquad \text{when } \sigma < 0 \end{aligned} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots [1.7]$$

Conditions must be determined for coefficients of system [1.2] upon whose satisfaction the trivial solution of the system [1.2] would be stable on the whole for any continuous function, satisfying conditions in [1.7].

It will be shown that *if* $\det A$ *is unequal to* 0 *and matrix* [1.5] *has zero characteristic values for* $\mu = \mu_0$, *the problem reduces to the original one.*

In material which follows (see Eqs. [3.2 and 3.5]), we obtain the formula

$$\det A_{\mu_0{}^{(0)}} = -\mu_0[\rho + (b, A^{-1}a)] \times \det A$$

Since according to conditions $\det A \neq 0$, $\det A_{\mu_0{}^{(0)}} = 0$, we have

$$\Gamma^2 \equiv \rho + (b, A^{-1}a) = 0 \qquad [1.8]$$

Assuming in the equations of [1.2] that $\varphi_1(\sigma) = \varphi(\sigma) - \mu_0\sigma$, $x = z + k\sigma$, and selecting the constant vector k so that components containing the factor σ vanish from the right-hand sides of the resultant equations (this can be done by virtue of relationship [1.8]), we obtain $k = -\mu_0 A^{-1}a$

$$\begin{aligned} dz/dt &= A_1z + a_1\varphi_1(\sigma) \\ d\sigma/dt &= b^*z - \rho\varphi_1(\sigma) \end{aligned} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots [1.9]$$

where

$$\begin{aligned} A_1 &= A - kb^* \\ a_1 &= a + \rho k \end{aligned}$$

and functions $\varphi_1(\sigma)$ satisfy the relationships in [1.4].

If we set $\varphi_1(\sigma) = 0$ in [1.9] we obtain a linear system, obtained by a nonsingular linear transformation from system [1.2], in which $\varphi(\sigma) = \mu_0\sigma$. Therefore, the characteristic values of matrix $A_{\mu_0{}^{(0)}}$ conform with the characteristic values of matrix A_1 less one zero proper value.

Thus *if* $\det A$ *is unequal to* 0, *and on the boundary of zone of stability* (μ_0, ∞) *the matrix of coefficients of linearized system* [1.2] $[\varphi(\sigma) = \mu_0\sigma]$ *has only zero characteristic values, the more general problem, when conditions of* [1.7] *are applied to servomotor performance, reduces to the original one.*[7]

One can proceed in another way. Assuming

$$x_1 = \begin{pmatrix} x \\ \sigma \end{pmatrix} \qquad \varphi_1(\sigma) = \varphi(\sigma) - \mu_0\sigma$$

we write the system [1.2] in the form

$$\begin{aligned} dx_1/dt &= Qx_1 + q\varphi_1(\sigma) \\ \sigma &= c^*x \end{aligned} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots [1.10]$$

where

$$c = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$q = \begin{pmatrix} a \\ -\rho \end{pmatrix}$$

$$Q = A_{\mu_0{}^{(0)}}$$

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[4] There is also some interest in the case where $\varphi'(\sigma) = \infty$ at some points. Therefore we do not require that $\varphi(\sigma)$ satisfy Lipshits' conditions and cannot guarantee the uniqueness of the solution.

[5] We see in material which follows that for the considered range of problems this condition is always satisfied. In general, solution of the system [1.1], with the initial conditions y_0, ξ_0, $\sigma_0 = b^*y_0 - \rho\xi_0$, is determined from formulas

$$y = \int_0^t x dt + y_0 \qquad \xi = \xi_0 + \int_0^t \varphi(\sigma) dt$$

where x and σ are solutions to the system [1.2] with initial conditions σ_0, $x_0 = Ay_0 - a\xi_0$.

[6] One may also assume that other zones of stability exist as well but we are considering the zone (μ_0, ∞).

[7] If when $\mu = \mu_0$ the matrix $A_{\mu_0{}^{(0)}}$ has zero and purely imaginary characteristic values, then in the system [1.9] the matrix A_1 would have purely imaginary values. We cannot consider this case here.

A system of the form [1.10], a system of equations, "directly automatically controlled," is generally speaking more general than system [1.2]. A particular case of the system [1.10] is the "Aiserman type" system which differs from [1.2] by the presence of items with multiples of σ in the right-hand sides.

It can be pointed out that these systems, with the presence of zones of stability of the form (μ_0, ∞) in the linear systems, when there are zero roots on the boundary of the stability zone (when $\mu = \mu_0$), also lead to systems of the form [1.2], where functions $\varphi(\sigma)$ satisfy conditions in [1.4]. We will not linger over that now.

Section 2 A. I. Lurye's Method. Resolving Equations. Stability Theorems

1′ For system [1.2] we will seek the Lyapunov function in the form

$$V(x, \sigma) = (Hx, x) + \int_0^\sigma \varphi(\sigma)d\sigma \qquad H^* = H \qquad [2.1]$$

Differentiation along solutions of system [1.2] provides

$$\dot{V} = -W(x, \sigma) = -(x^*Gx + 2g^*x\varphi + \rho\varphi^2) \qquad [2.2]$$

where

$$\begin{aligned} -G &= A^*H + HA \\ -g &= Ha + \tfrac{1}{2}b \end{aligned} \qquad [2.3]$$

THEOREM 1 *Let us assume that in system* [1.2]:

1 *Matrix A has characteristic values with negative real parts.*

2 *Function $\varphi(\sigma)$ is continuous and satisfies conditions in* [1.4].

3 *It is possible to find the Lyapunov function V in the form* [2.1], *where $(Hx, x) > 0$ when $x \neq 0$, whose derivative, calculated from system* [1.2], *would be negatively definite, $\dot{V} = -W < 0$ for $\|x\| + |\sigma| \neq 0$.*

Then, the trivial solution for system [1.2] *is stable on the whole.*

The statement in the hypothesis that there is a divergence of integrals

$$\int_{-\infty}^0 \varphi(\sigma)d\sigma \qquad \int_0^\infty \varphi(\sigma)d\sigma$$

follows from the theorems of Barbashin and Krasovskii covering overall stability (8) [see (9) also]. The theorem was proved in general in (10).[8] For $n = 2$, with some additional hypotheses, this theorem was proved earlier by Skachkov.

Let us determine the condition where $\dot{V} = -W < 0$ when $\|x\| + |\sigma| \neq 0$. Since $W(0, \sigma) = \rho\varphi^2$, a necessary condition would be the inequality $\rho > 0$ which was realized according to the hypothesis.

We show that for the positiveness of function $W(x, \sigma)$, it is necessary and sufficient that

$$\rho G - gg^* > 0 \qquad [2.3']$$

From equality [2.2] we have

$$W = x^*\left(G - \frac{1}{\rho}gg^*\right)x + \rho\left(\varphi + \frac{1}{\rho}g^*x\right)^2 \qquad [2.4]$$

from which the sufficiency of conditions [2.3′] follows. Let us prove the necessity. Assume that conditions of [2.3] are not realized. Then, for some vector $x_0 \neq 0$, $x_0^*[G - (1/\rho)gg^*]x_0 \leq 0$.

Assume that $g^*x_0 \neq 0$. For any fixed $\sigma = \sigma_0$ let us determine λ from condition $\sqrt{\rho}\varphi(\sigma_0) + (\lambda/\sqrt{\rho})g^*x = 0$. Then $W(\lambda x_0, \sigma_0) \leq 0$. Conflict with the positiveness of the function $W(x, \sigma)$ shows that this case was impossible.

Let $g^*x_0 = 0$. Then $W(x_0, 0) = x_0^*Gx_0 \leq 0$, which is also impossible according to the hypothesis. Thus, the necessity of conditions in [2.3′] are proved.

Condition [2.3′] can be written in the form

$$-\rho(A^*H + HA) - (Ha + \tfrac{1}{2}b)(a^*H + \tfrac{1}{2}b^*) = -(\rho/2)C > 0 \qquad [2.5]$$

where C is some constant positively definite matrix. The factor $(\rho/2)$ was supplied for the convenience of further computations.

Equation [2.5] or

$$Haa^*H + B^*H + HB + \tfrac{1}{4}bb^* + (\rho/2)C = 0 \qquad [2.6]$$

where

$$B = \rho A + \tfrac{1}{2}ab^* \qquad [2.7]$$

is the quadratic equation for matrix C, with a definite matrix H.

2′ By using Theorem 1, it is possible to solve the problem formulated in Section 1 in the following manner.

Take any matrix $H > 0$ and from formulas [2.5] or [2.6] and [2.7] we determine matrix C. Matrix C would depend on the parameter coefficients of the assigned system of equations in [1.2].

Condition $C > 0$ or the positiveness of the main minors of matrix C also provides conditions for coefficients, whose completion according to Theorem 1, would show overall stability. These sufficient conditions would be more or less broad depending on selection of matrix H.

Our problem is obtaining conditions which would embrace all similar sufficient conditions.

3′ LEMMA 1 *If all characteristic values of matrix K have negative real parts, then the solution for X in the equation*

$$K^*X + XK = -C \qquad [2.8]$$

is unique and determined by the formula

$$X = \int_0^\infty e^{K^*t}Ce^{Kt}dt \qquad [2.9]$$

If $C > 0$ then $X > 0$; if $C \geq 0$ then $X \geq 0$.

If in Equation [2.8] $C > 0$ *and it is known that $X \geq 0$, then the characteristic values of matrix K have negative real parts and $X > 0$.*

The lemma statement is a partial case (a slight extension) of the well-known Lyapunov theorem, the simple proof of which is in (12), pp. 430–431. Formula [2.9] is checked by substitution in [2.8] and by partial integration. It follows from [2.9] that for any vector a

$$(Xa, a) = \int_0^\infty (Cz, z)dt \qquad \text{where } z = e^{Kt}a$$

Hence $(Xa, a) \geq 0$, if $C \geq 0$ and $(Xa, a) > 0$ if $C > 0$.

4′ LEMMA 2 *Suppose that:*

1 *Equations* [2.5] *or* [2.6] *and* [2.7] *are realized where $C > 0$, and H is some symmetrical matrix.*

2 *For some ν, $0 \leq \nu \leq 1$, the matrix*

$$K_\nu = \rho A + \nu ab^*$$

has all characteristic values with negative real parts.

Then $H > 0$ and matrix K_ν, for any ν, $0 \leq \nu \leq 1$, has characteristic values with negative real parts.

Proof. For any μ

$$C_\mu \equiv \frac{\rho}{2}C + (\mu Ha + \tfrac{1}{2}b)(\mu Ha + \tfrac{1}{2}b)^* > 0$$

Substituting the value C from Equation [2.5], we obtain

$$K_\nu^*H + HK_\nu = -C_\mu - (1 - \mu^2)Haa^*H \qquad [2.10]$$

[8] We note that, generally speaking, the graph of function $\varphi(\sigma)$ can have a vertical tangent at some points, i.e., conditions of the theorem uniqueness (11) may not be satisfied. In addition, the theorem does not a priori propose extending the solutions when $t \to \infty$; this follows from the theorem statement and is proved. It is possible to discard condition 1 in formulation of the theorem; it follows from condition 3. With condition 1 present it is possible to discard the condition $(Hx, x) > 0$, $x \neq 0$; it follows from the remaining conditions.

where $\nu = \frac{1}{2}(1 - \mu)$. The left-hand part of Equation [2.10] was negative when $|\mu| \leq 1$, i.e., when $0 \leq \nu \leq 1$. Using Lemma 1, we obtain confirmation of Lemma 2.

5′ Henceforth, down to Section 4, it will be assumed that the characteristic values of matrix A have negative real parts. It follows from Theorem 1 and Lemma 2 that for overall stability, it is sufficient for us to prove that for some matrix $C > 0$ the quadratic equation [2.5] or [2.6] has a symmetrical and real matrix H^* as a solution.[9]

We can consider the matrix equation [2.5] or [2.6] as the system of $[n(n + 1)]/2$ scalar quadratic equations for the same number of unknowns—elements of matrix H. These equations contain $[n(n + 1)]/2$ parameters—elements of matrix $C > 0$. We require that this system have real solutions for some parameter values.

We now reduce this system to a system of n quadratic equations, in n unknowns, containing n parameters.

We determine the linear operator $Y = L(X)$ by the formula

$$A^*Y + YA = -X \qquad [2.11]$$

According to Lemma 1

$$L(X) = \int_0^\infty e^{A^*t} X e^{At} dt$$

We denote

$$-u = Ha + \tfrac{1}{2}b \qquad [2.12]$$

From relationship [2.5], we have

$$\rho(A^*H + HA) = -(\rho/2)C - uu^* \qquad [2.13]$$

$$\rho H = L(uu^*) + \tfrac{1}{2}\rho L(C) \qquad [2.14]$$

Substituting Equation [2.12], we obtain an equation for u

$$L(uu^*)a + \rho u + \tfrac{1}{2}\rho(b + c) = 0 \qquad [2.15]$$

where

$$c = L(C)a \qquad [2.16]$$

Equation [2.15] is written in the vector form with the system of n quadratic equations relative to components of vector u, containing n parameters—the components of vector c.

Equation [2.15] can now be simplified by special selection of matrix C.

Let us assume $C = \epsilon I$, $\epsilon > 0$ and proceed to the point when $\epsilon \to 0$. We obtain the equation

$$L(uu^*)a + \rho u + \tfrac{1}{2}\rho b = 0 \qquad [2.17]$$

Henceforth, we will find it convenient to write equations [2.15, 2.16 and 2.17] in another form.

Designating $U = L(uu^*)$, $T = L(C)$, instead of [2.15 and 2.16] we obtain the system

$$\begin{aligned} &\text{(I)} \quad A^*T + TA = -C \\ &\text{(II)} \quad c = Ta \\ &\text{(III)} \quad A^*U + UA = -uu^* \\ &\text{(IV)} \quad Ua + \rho u \tfrac{1}{2}\rho(b + c) = 0 \end{aligned} \qquad [2.18]$$

and instead of [2.17], the system

$$\begin{aligned} &\text{(I)} \quad A^*U + UA = -uu^* \\ &\text{(II)} \quad Ua + \rho u + \tfrac{1}{2}\rho b = 0 \end{aligned} \qquad [2.19]$$

THEOREM 2 *Let us assume that for some ν, $0 \leq \nu \leq 1$, the characteristic values of matrix $K_\nu = \rho A + \nu ab^*$ have negative real parts.*

1 *If Equation* [2.17] *or* [2.19], (I), (II) *allows a real solution both for the assigned vector of b and all vectors of b, sufficiently close to that assigned, then the trivial solution of system* [1.2] *is stable on the whole for any continuous function $\varphi(\sigma)$, satisfying conditions in* [1.4].

2 *Let C be the arbitrary positively definite matrix and vector c be determined by formulas* [2.18], (I) *and* (II). *The statement in Point* 1 *is correct if Equation* [2.15] *or Equations* [2.18], (III) *and* (IV) *have a real solution u.*

Proof. First, let us prove point 2. Let u be the real solution of Equation [2.15] for some matrix $C > 0$. Inasmuch as $L(X)^* = L(X^*)$, the matrix H, determined by formula [2.14], would be symmetrical; in addition [2.12] would be realized. Equation [2.13] follows from [2.14]; using [2.12], we obtain equality [2.5]. The statement of Theorem 2 follows from Theorem 1 and Lemma 2.

Point 1 follows from 2, inasmuch as it is possible to take $C = \epsilon I$ as C where $\epsilon > 0$ is a sufficiently small number.

Note, that if the theorem conditions are satisfied, then according to Lemma 2, matrices K_ν for any ν, $0 \leq \nu \leq 1$ (matrix A, in particular), have characteristic values in the left semiplane. In material which follows, we show that this condition for matrix $K_1 = \rho A + ab^*$ means, speaking somewhat inexactly, that when $\varphi(\sigma) = \mu\sigma$, with a sufficiently large μ (a rather "high speed" servomotor), the system [1.2] is asymptotically stable.

The following observation also follows from the preceding. *Assume that the resultant conditions, connecting elements of matrix A and vectors a and b, whose satisfaction indicates that the equation* [2.15] *or* [2.18], (IV) *has a real solution and that for some vector $c = L(C)a$, $C > 0$. Then, these conditions would be sufficient for overall stability, embracing all sufficient conditions which can be obtained with the aid of the form* [2.1] *of the Lyapunov function.*

Equations [2.18], (III) and (IV) coincide in form with equations in [2.19]. Following Lurye, we will call the equations in [2.19] and [2.18], (III) and (IV), *resolving equations.*

6′ We show that point 1 of Theorem 2 essentially coincides with Lurye's theorem (1), p. 68. Let us assume, as in (1), that matrix A is reduced to a diagonal form and at first, for simplicity, that all characteristic values λ_j of matrix A are real.

We designate $u = \|\xi_j\|^n_{j=1}$, $U = \|\eta_{ij}\|^n_{ij=1}$. On a basis where A is a diagonal matrix and all components of vector a equal a unit,[10] $b = \|\beta_j\|^n_{j=1}$, Equation [2.19] takes on the form

$$(\lambda_j + \lambda_i)\eta_{ij} = -\xi_i\xi_j \qquad \sum_{j=1}^{n} \eta_{ij} + \rho\xi_i + \frac{1}{2}\rho\beta_i = 0$$

or

$$-\xi_i \sum_{j=1}^{n} \frac{\xi_j}{\lambda_i + \lambda_j} + \rho\xi_i + \frac{1}{2}\rho\beta_i = 0 \qquad i = 1, \ldots, n \qquad [2.20]$$

Substitution of $\xi_j = \sqrt{\rho}a_j$ reduces this system to system [3.7], p. 68 (1).

Inasmuch as the conversion from the assigned basis to the basis where A is a diagonal is effected by means of a real matrix, the solution of ξ_j in system [2.20] should be real.

Now, let some of the λ_j be complex. Designate ξ_j' as components of vector u in the original basis, ξ_j in the basis where A is diagonal. Then

$$\xi_j' = \sum_{i=1}^{n} \pi_{ji}\xi_i$$

[9] It would have been interesting to solve the matrix quadratic equation [2.5] in a general form. We note that equation $XAX^* + XB + B^*X^* + C = 0$ (where A, generally speaking, is a special matrix) can be solved explicitly. But separating the partial solution, satisfying condition $X = X^*$, from the total solution leads again to equations of the type [2.5].

[10] It is easy to show, following Lurye, that the substitution $x = Sx_1$ exists, reducing system [1.2] to the indicated form. Columns of matrix S are characteristic vectors of matrix A, normalized in the appropriate manner. It is also evident that the requirement that all components of vector a be equal to unity is not necessary.

where $p_i = \|\pi_{ji}\|^n_{j=1}$ are vectors of matrix A.

Let $\bar{\lambda}_1 = \lambda_2, \ldots, \bar{p}_1 = p_2, \ldots$. In order that ξ_1', ξ_2' be real, it is sufficient to require that ξ_1 and ξ_2 and, in general, ξ_{j_1} and ξ_{j_2}, corresponding to complex conjugates λ_{j_1} and λ_{j_2}, be complex conjugate.

We obtained, in full, the theorem of Lurye (1), p. 68.

We note that all the preceding can be extended, without change, to the case where the coefficients of system [1.2] were complex. In this case

$$V = x^*Hx + \int_0^\sigma |\varphi(\sigma)|^2 d\sigma$$

$H = H^*$ and the operation of "*" denotes, apart from transposition, the conversion to complex conjugate values. With this observation, Theorem 2, Equations [2.15 and 2.17] remain without change. Equations of [2.20] relative to complex ξ_j take on the form

$$-\xi_j \sum_{i=1}^{n} \frac{\bar{\xi}_i}{\lambda_j + \bar{\lambda}_i} + \rho\xi_j + \frac{1}{2}\rho\beta_j = 0 \qquad j = 1, \ldots, n$$

7′ Let us return once more to the quadratic inequality in [2.5] for matrix H. This inequality does not depend on function $\varphi(\sigma)$. Therefore, if the Lyapunov function of the form [2.1] does exist for some function $\varphi(\sigma)$, satisfying conditions in [1.8], then it also exists for any other such function, and according to Theorem 1 of system [1.2] is stable on the whole. Noting that the Lyapunov function is determined with an accuracy to a positive multiplication constant, we obtain the following theorem.

THEOREM 3 *In order that the trivial solution of system* [1.2] *be stable on the whole for any function $\varphi(\sigma)$, satisfying conditions in* [1.4], *it is sufficient (and in order to detect this stability with the aid of the Lyapunov function* [2.1] *it is even necessary) that the linear system, obtained from* [1.2] *when $\varphi(\sigma) = \mu\sigma$, where μ is any fixed number larger than* 0, *have a Lyapunov function of the form*

$$V(x, \sigma) = (Hx, x) + \sigma^2 \qquad H > 0 \qquad [2.21]$$

For comparison, we note that any linear system [1.2], for which the trivial solution is asymptotically constant, has a Lyapunov function of the form

$$V(x, \sigma) = (Hx, x) + 2(h, x) \times \sigma + \sigma^2 \qquad H > hh^*$$

It is easy to conclude that the problem of finding a Lyapunov function [2.21] for a linear system is equivalent to the following problem. The conditions have to be determined which are imposed on the elements of matrix A and vector c so that the system

$$A^*X + XA = -U \qquad Xc = c \qquad [2.22]$$

should admit solutions in the form of symmetrical matrices X, $U > 0$. Here A is the matrix [1.5]; it has characteristic values with negative real parts.

We also note that the Equations [2.18], (I) and (II) can be reduced to the form [2.22] and the reverse.

8′ It is evident that it is easier to check Point 1 of Theorem 2 than Point 2. An advantage of Point 2 is that it furnishes all sufficient conditions which can be obtained with the aid of the Lyapunov function, form [2.1]. Let us analyze, in more detail, the equations in [2.18], appearing in Point 2. The last two equations in [2.18] coincide with the equations in [2.19] of Point 1′, if vector b is replaced by $b + c$. The first two equations in [2.18] serve to determine those conditions which should be satisfied by the elements of vector c.

We will call the elements of vector c *additional parameters.* As was indicated in the foregoing, the complexity of algebraic computations for systems of a larger order are explained, to a considerable extent, by the large number of unknown parameters. In investigating the equations in [2.19] we still have n "excess" additional parameters; in addition, still other algebraic computations are added which are connected with the investigation of conditions, to be satisfied by these additional parameters. In this connection, it seems very tempting to make the following hypothesis: *Point 1 of Theorem 2 furnishes the same sufficient conditions as does Point 2* or, in other words, *if Equations* [2.18] (II) *and* (IV) *have solutions for some values of the additional parameters, then they also have solutions when the additional parameters are arbitrary, sufficiently small numbers.*

The correctness of this hypothesis for $n = 1$ is quickly established. In this case the number $A < 0$; from the first two equations in [2.18], we derive that c is an arbitrary number, having the sign of a. (Where $a = 0$ it was trivial for any n.) From the last two equations in [2.18] we have

$$u = \frac{(-A)}{a}\left[-\rho \pm \sqrt{\rho^2 - \rho\frac{ab}{(-A)} - \rho\frac{ac}{(-A)}}\right]$$

If the radicand expression is positive for some values of c of the indicated form, then

$$\rho^2 - \rho\frac{ab}{(-A)} > 0 \qquad [2.23]$$

which also indicates the correctness of our statement.

The condition of [2.23], i.e., the condition

$$\Gamma^2 \equiv \rho + A^{-1}ab > 0 \qquad [2.24]$$

is sufficient (and in material which follows, we show that it is even necessary) as a condition for system [1.2], when $n = 1$, to be stable on the whole for any function $\varphi(\sigma)$ of the indicated form.

We will prove the correctness of our hypothesis for $n = 2$.

THEOREM 4 *Assume that:*

1 $n = 2$.

2 *For some ν, $0 \leq \nu \leq 1$, the matrix*

$$K_\nu = \rho A + \nu ab^*$$

has all the characteristic values in the left semiplane.

3 *The quadratic equation* [2.5] (or [2.6]), *relative to the symmetrical matrix H, has a real solution for some matrix $C >$ 0.*

Then, $\epsilon_0 > 0$ is found so that Equation [2.5] (*or* [2.6]) *would have a real solution H for any matrix C, so that $0 < C \leq \epsilon_0 J$.*

Proof.[11] Rejecting the trivial case, we immediately assume that a is not equal to 0. Equation [2.5] or [2.6] is equivalent, as was shown in the foregoing, to equations in [2.18]. Let us first consider Equation [2.19] instead of the last equations in [2.18]. According to Lemma 2, matrix K_ν, for any ν, $0 \leq \nu \leq 1$, especially matrix $A = K_0$ has a spectrum in the left semiplane. Therefore, A^{-1} exists.

Assume, first, that vector a is not a characteristic vector of matrix A. Then, vectors a and $a_1 = A^{-1}a$ would be linearly independent. Let us designate

$$S = \|a|a_1\|$$

the matrix with columns a and a_1 and in the system make the substitution

$$x = Sx'$$

We obtain a system of the same form

$$\frac{dx'}{dt} = A'x' + a'\varphi(\sigma) \qquad \frac{d\sigma}{dt} = (b', x') - \rho\varphi(\sigma) \qquad [2.25]$$

where

$$A' = S^{-1}AS \qquad a' = S^{-1}a \qquad b' = S^*b$$

[11] We develop the proof in such a way as to make a simultaneous illustration of the Lurye method in our exposition and to obtain sufficient stability conditions for $n = 2$.

Let

$$Aa = -\alpha_1 a - \alpha_2 a_1 \qquad [2.26]$$

$$\beta_1 = (a, b) \qquad \beta_2 = (A^{-1}a, b) \qquad [2.27]$$

Then

$$A' = \begin{pmatrix} -\alpha_1 & 1 \\ -\alpha_2 & 0 \end{pmatrix} \qquad b' = \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} \qquad a' = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad [2.28]$$

The characteristic equation

$$\det (A - \lambda I) \equiv \det (A' - \lambda I) = 0$$

has the form

$$\det (A - \lambda I) \equiv \lambda^2 + \alpha_1\lambda + \alpha_2 = 0$$

and since the roots of this equation lie in the left semiplane, we have

$$\alpha_1 > 0 \qquad \alpha_2 > 0$$

Let us designate

$$u = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} \qquad U = \begin{pmatrix} \varphi_{11} & \varphi_{12} \\ \varphi_{21} & \varphi_{22} \end{pmatrix} \qquad \varphi_{12} = \varphi_{21}$$

Equation [2.19] (II) takes on the form

$$\begin{pmatrix} \varphi_{11} \\ \varphi_{21} \end{pmatrix} + \rho\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} + \frac{\rho}{2}\begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} = 0 \qquad [2.29]$$

We find

$$UA = \begin{pmatrix} -\alpha_1\varphi_{11} - \alpha_2\varphi_{12} & \varphi_{11} \\ -\alpha_1\varphi_{21} - \alpha_2\varphi_{22} & \varphi_{21} \end{pmatrix}$$

Considering that $A^*U = (UA)^*$, we write Equation [2.19], (1) in the form

$$\begin{pmatrix} -2\alpha_1\varphi_{11} - 2\alpha_2\varphi_{12} & * \\ \varphi_{11} - \alpha_1\varphi_{21} - \alpha_2\varphi_{22} & 2\varphi_{21} \end{pmatrix} = -\begin{pmatrix} \xi_2{}^1 & \xi_1 & \xi_2 \\ \xi_1 & \xi_2 & \xi_0{}^2 \end{pmatrix}$$

Hence[12]

$$\varphi_{12} = -\frac{1}{2}\xi_2{}^2 \qquad \varphi_{11} = \frac{1}{2\alpha_1}\xi_1{}^2 + \frac{\alpha_2}{2\alpha_1}\xi_2{}^2 \qquad [2.30]$$

Substituting these values in Equation [2.29], we obtain the following resolving equations

$$\xi_1{}^2 + \alpha_2\xi_2{}^2 + 2\alpha_1\rho\xi_1 + \rho\alpha_1\beta_1 = 0$$
$$\xi_2{}^2 - 2\rho\xi_2 - \rho\beta_2 = 0$$

Solving them in succession, we find the conditions (necessary and sufficient) for the existence of real solutions ξ_1, ξ_2

$$\Gamma^2 \equiv \rho + \beta_2 \geqslant 0$$
$$D^2 \equiv \rho\alpha_1{}^2 - \alpha_1\beta_1 - \alpha_2(\sqrt{\rho} - \Gamma)^2 \geqslant 0 \qquad [2.31]$$

According to Point 1 of Theorem 2, fulfillment of these conditions for numbers β_1, β_2, arbitrarily close to the assigned ones, indicates the overall stability of system [1.2]. Thus, conditions

$$\Gamma^2 \equiv \rho + \beta_2 > 0$$
$$D^2 \equiv \rho\alpha_1{}^2 - \alpha_1\beta_1 - \alpha_2(\sqrt{\rho} - \Gamma)^2 > 0 \qquad [2.32]$$

are sufficient conditions for the stability of system [1.2].

As follows from the preceding, these conditions indicate the solvability of quadratic equation [2.5] (or [2.6]) for any sufficiently small matrix $C > 0$.

The solution of this equation for the fixed matrix $C > 0$ was equivalent to the existence of a real solution u in equations of [2.19].

Designating γ_1, $(-\gamma_2)$ the components of vector c and comparing Equations [2.18], (III) and (IV) with the equations in [2.19], we obtain from [2.31] the following conditions for the solution of Equations [2.19], (III) and (IV)

$$\tilde{\Gamma}^2 \equiv \rho + \beta_2 - \gamma_2 \geqslant 0$$
$$\tilde{D}^2 \equiv \rho\alpha_1{}^2 - \alpha_1(\beta_1 + \gamma_1) - \alpha^2(\sqrt{\rho} - \tilde{\Gamma})^2 \geqslant 0 \qquad [2.33]$$

Let us find the conditions satisfied by the additional parameters γ_1, γ_2

Let

$$T = \begin{pmatrix} \psi_{11} & \psi_{12} \\ \psi_{12} & \psi_{22} \end{pmatrix} \qquad C = \begin{pmatrix} \zeta_{11} & \zeta_{12} \\ \zeta_{12} & \zeta_{22} \end{pmatrix}$$
$$\zeta_{11} > 0 \qquad \zeta_{22} > 0 \qquad \zeta_{11}\zeta_{22} - \zeta_{12}{}^2 > 0$$

Comparing the first equations in [2.18 and 2.19] we find, from Equation [2.30]

$$\gamma_1 = \psi_{11} = \frac{1}{2\alpha_1}\zeta_{11} + \frac{\alpha_2}{2\alpha_1}\zeta_{22}$$
$$\gamma_2 = -\psi_{22} = \tfrac{1}{2}\zeta_{22}$$

ζ_{11}, ζ_{22} are arbitrary positive numbers. Therefore, the numbers γ_1, γ_2 are arbitrary provided they satisfy inequalities

$$\gamma_1 > (\alpha_2/\alpha_1)\gamma_2 > 0 \qquad [2.34]$$

We have, from relationships [2.31 and 2.33]

$$\tilde{\Gamma}^2 = \Gamma^2 - \gamma_2$$
$$\tilde{D}^2 = D^2 - \alpha_1\gamma_1 - \alpha_2[(\sqrt{\rho} - \tilde{\Gamma})^2 - (\sqrt{\rho} - \Gamma)^2] \qquad [2.35]$$

We must show that the inequalities in [2.32] follow from the inequalities of [2.33].

Evidently, the first inequality was realized.

Using Equation [2.34], we have

$$D^2 = \tilde{D}^2 + \alpha_1\gamma_1 + \alpha_2[(\tilde{\Gamma} - \sqrt{\rho})^2 - (\Gamma - \sqrt{\rho})^2] \geqslant \tilde{D}^2 + \alpha_2\gamma_2 +$$
$$\alpha_2[(\tilde{\Gamma} - \sqrt{\rho})^2 - (\Gamma - \sqrt{\rho})^2] = \tilde{D}^2 + 2\alpha_2\sqrt{\rho}(\Gamma - \tilde{\Gamma}) > \tilde{D}^2 \geqslant 0$$

and our statement is proved.

It remains for us to consider the case when

$$Aa = -\alpha a \qquad \alpha > 0$$

In material which follows, we show that in this case for any n the solvability conditions for equations in [2.18] have the form

$$\tilde{\Gamma}^2 \equiv \rho - \frac{1}{\alpha}(b, a) - \frac{1}{\alpha}(c, a) \geqslant 0$$

Since $T > 0$ then $(c, a) = (Ta, a) > 0$ and

$$\Gamma^2 \equiv \rho - (1/\alpha)(b, a) \geqslant \tilde{\Gamma}^2$$

i.e., $\Gamma^2 > 0$, which is the solvability condition for the equations in [2.19] (for vectors of b, sufficiently close to that assigned). Thus, in this case the theorem's statement is correct for any n. The theorem is proved.

Simultaneously, we obtained the following statement.

System [1.2] *is considered for* $n = 2$ *on the assumption that the equation*

$$\det (A - \lambda I) \equiv \lambda^2 + \alpha_1\lambda + \alpha_2 = 0$$

has roots in the left semiplane ($\alpha_1 > 0$, $\alpha_2 > 0$). *We designate*

[12] We don't find φ_{22}, since it is evident from the relationships in Equation [2.29] that we do not need φ_{22}.

$\beta_1 = (a, b)$, $\beta_2 = (A^{-1}a, b)$. *If the conditions in* [2.32] *are satisfied, then system* [1.2] *is stable on the whole.*

These conditions were equivalent to conditions in (1) (although these conditions are easier to check). We also showed that this is the most general condition which can be obtained by considering the Lyapunov function in form [2.1].

It is easy to show that when $\beta_2 \geqslant 0$ they are also necessary for stability with *any* function $\varphi(\sigma)$ of the form [1.4], namely, that they correspond to conditions, obtained for function $\varphi(\sigma) = \mu\sigma$, where $\mu > 0$, i.e., for any given case the Aiserman problem is solved positively.

In Popov's work (13) this result was considerably amplified: With the exception of some cases, for some system of [1.2], $n = 2$, the Aiserman problem is resolved positively even for $\beta_1 < 0$. Popov used Lurye's method (1). In a thesis by Degen, it was shown that with the use of Popov's idea and Lurye's method, in the form shown here, for system [1.2], $n = 2$, the Aiserman problem is always solved positively.[13]

We also note the book by Pliss (15) where, for a wide class of systems for $n = 2$, final results are obtained in a concrete form.

Section 3 Resolving Equations

1′ Following Lurye (1), we derive some necessary conditions for the existence of solutions to Equations [2.19] or [2.18], (III) and (IV).

Assume, as in Section 2, that det A is unequal to 0.

Multiplying the first equation in [2.19] from the left by $a^*.1^{*-1}$ and from the right by $.1a^{-1}$, we obtain

$$2(Ua, .1a^{-1}) = -(u, .1a^{-1})^2$$

Designating

$$\zeta = \frac{1}{\sqrt{\rho}}\,(u, .1a^{-1}) \qquad [3.1]$$

and substituting the value Ua from the second equation in [2.19], we obtain

$$\zeta^2 - 2\sqrt{\rho}\,\zeta - (b, .1a^{-1}) = 0$$

A necessary condition for the existence of a real ζ for the assigned b and all b, sufficiently close to that assigned, would be

$$\Gamma^2 = \rho + (b, A^{-1}a) > 0 \qquad [3.2]$$

In addition

$$\zeta = \sqrt{\rho} \pm \Gamma \qquad [3.3]$$

We show that the condition $\Gamma^2 > 0$ is necessary as well for overall stability.[14] Expanding the determinant of matrix [1.3] according to the last column, we obtain

$$\det \begin{pmatrix} A & a \\ b^* & -\rho \end{pmatrix} = -\rho\Delta - (Qa, b) \qquad [3.4]$$

where $\Delta = \det A$ and Q is the transposed matrix of the algebraic supplements to matrix A. If $\Delta \neq 0$, then $Q = \Delta A^{-1}$, and according to formula [3.4] the determinant of matrix [1.5] equalled

$$\det \begin{pmatrix} A & a\mu \\ b^* & -\rho\mu \end{pmatrix} = -\Delta\Gamma^2\mu \qquad [3.5]$$

But for real matrices, in which the real characteristic values were negative, the determinant sign equalled $(-1)^n$, where n is the order of the matrix. Since the characteristic values of matrix [1.5], when $\mu > 0$, and matrix A satisfy this condition, it therefore follows from [3.5] that $\Gamma^2 > 0$.

Still another meaning can be imparted to condition $\Gamma^2 > 0$. We designate by $\lambda_j(\mu)$, $j = 0, 1, \ldots, n$ characteristic values of matrix [1.5]. Where they have a sufficiently small μ, let n of them, $\lambda_1(\mu), \ldots, \lambda_n(\mu)$, be as close as desired to the characteristic values of matrix A and by definition nonzero. Then $\lambda_0(0) = 0$ and as the algebraic function $\lambda_0(\mu)$ permits the expansion

$$\lambda_0(\mu) = \delta_0\mu^{p/q} + \delta_1\mu^{p+1/q} + \ldots$$

where p and q are some whole numbers. Using [3.5], we obtain the identity in μ

$$-\Delta\Gamma^2\mu \equiv [\delta_0\mu^{p/q} + \ldots] \times \prod_{j=1}^{n} \lambda_j(\mu)$$

Since

$$\prod_{j=1}^{n} \lambda_j(0) = \Delta$$

then $p/q = 1$ and $\delta_0 = -\Gamma^2$. The expansion for $\lambda_0(\mu)$ thus has the form[15]

$$\lambda_0(\mu) = -\mu\Gamma^2 + \ldots \qquad [3.6]$$

The condition $\Gamma^2 > 0$ thus indicates that with a small $\mu > 0$

$$\lambda_0(\mu) < 0$$

2′ We now derive the condition that matrix [1.5] has characteristic values with negative real parts for all sufficiently large $\mu > 0$.[16]

Making a corresponding substitution in Equation [3.4] we find that the characteristic equation for the linear system [1.2] has the form

$$\det \begin{pmatrix} A - \lambda I & \mu a \\ b^* & -\mu\rho \end{pmatrix} \equiv \lambda\Delta(\lambda) - \mu\{\rho\Delta(\lambda) + [Q(\lambda)a, b]\} = 0 \qquad [3.7]$$

Here $\Delta(\lambda) = \det(A - \lambda I)$ and $Q(\lambda)$ is the transposed matrix of algebraic complements of matrix $A - \lambda I$. Substituting $\mu = 1/\epsilon$ and assuming that $\epsilon = 0$, we obtain the equation

$$\rho\Delta(\lambda) + [Q(\lambda)a, b] = 0 \qquad [3.8]$$

Let $\lambda_j^0 (j = 1, \ldots, n)$ be the roots of this equation. Then the inequality

$$\mathrm{Re}\lambda_j^0 \leqslant 0 \qquad j = 1, \ldots, n \qquad [3.9]$$

is necessary, and inequality

$$\mathrm{Re}\lambda_j^0 < 0 \qquad j = 1, \ldots, n \qquad [3.10]$$

sufficient for matrix [1.5] to have characteristic values with negative real parts for all sufficiently large $\mu > 0$. The inequalities of [3.9] are, *thus necessary for the system of resolving equations to have real solutions.*[17]

We show that λ_j^0 are characteristic values of matrix

$$\frac{1}{\rho}K_1 = A + \frac{1}{\rho}ab^* \qquad [3.11]$$

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[13] Here, we speak of the Aiserman problem for zones of stability $(0, \sim)$ or, in other words, about the Letov problem (2). This statement was also proved in a work by V. A. Pliss (Doklady Akad. Nauk SSSR, vol. 117, no. 2, 1957).

[14] This proof repeats almost literally an analogous proof by Lurye (1), p. 75. We present it here for completeness.

[15] It is easy to show that when $\Delta \neq 0$, $\lambda_0(\mu)$ can be expanded in integral powers of μ.

[16] We do not now exclude the case where det $A = 0$.

[17] Let us designate $\psi(\lambda) = \rho\Delta(\lambda) + [Q(\lambda)a, b]$. On the basis of general theorems on algebraic functions it follows that in the hypothesis $\psi'(\lambda_j^0) \neq 0$ the root $\lambda_j^0(\epsilon)$ of Equation [3.7] would be an analytical function of $\epsilon = 1/\mu$ around $\epsilon = 0$, $\lambda_j^0(\epsilon) = \lambda_j^0 + \epsilon\lambda_j^1 + \ldots$. In addition $\lambda_j^{(1)} = \lambda_j^0 \times \Delta(\lambda_j^0) \times [\psi'(\lambda_j^0)]^{-1}$. Hence, it is easy to conclude that when $\mathrm{Re}\lambda_j^0 = 0$, $\psi'(\lambda_j^0) \neq 0$ a necessary condition also would be

$$\omega_j = \mathrm{Im}\,[\Delta(\lambda_j^0)/\psi'(\lambda_j^0)] \geqslant 0$$

(see Point 2 of Theorem 4). Actually, expanding the determinant in powers of $1/\rho$, we obtain

$$\det\left(A - \lambda I + \frac{1}{\rho} ab^*\right) = \Delta(\lambda) + \frac{1}{\rho}\,[Q(\lambda)a,\, b] \qquad [3.11a]$$

Hence, it also follows that the characteristic equation

$$\det\left(\frac{1}{\rho} K_1 - \lambda I\right) = 0$$

coincides with Equation [3.8].

We also note the formula obtained from comparison of formulas [3.11a] ($\lambda = 0$) and [3.5]

$$\det\begin{pmatrix} A & a \\ b^* & -\rho \end{pmatrix} = -\rho \det\left(A^* + \frac{1}{\rho} ab^*\right) \qquad [3.12]$$

3′ Let us consider the resolving equations in [2.19]. Inasmuch as each of the components of vector Ua is a quadratic form of the components of vector $u = \|\xi_j\|_{j=1}^{n}$, Equation [2.19], (II) has the form

$$\sum_{i,j=1}^{n} \alpha_{ij}^{(m)}\, \xi_i \xi_j + \rho\xi_m + \frac{1}{2}\,\rho\beta_m = 0 \qquad m = 1, \ldots, n \qquad [3.13]$$

The system [3.13] should have real solutions $\xi_1, \ldots, \xi_n$ for the assigned $\beta_1, \ldots, \beta_n$ and for all $\beta_1, \ldots, \beta_n$, close enough to the assigned ones.

It follows from the general theory of elimination that, eliminating sequentially $\xi_n, \xi_{n-1}, \ldots$, it is possible to arrive, following the finite number of rational algebraic operations, at one equation

$$\eta(\xi) \equiv \gamma_0 \xi^N + \gamma_1 \xi^{N-1} + \ldots + \gamma_N = 0 \qquad [3.14]$$

In addition, $N \leq 2^n$ (or $N \leq 2^{n-1}$, where relationship [3.3] is used).[18]

In practice, the elimination of unknowns is carried out by the use of the Euclidean algorithm.

Separating the unknown ξ_n, we write the system [3.13] in the form

$$\begin{aligned} &\alpha_{nn}^{(1)}\xi_n^2 + \psi_1^{(1)}\xi_n + \psi_1^{(2)} = 0 \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ &\alpha_{nn}^{(n)}\xi_n^2 + \psi_n^{(1)}\xi_n + \psi_n^{(2)} = 0 \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \end{aligned} \qquad [3.15]$$

where $\psi_j^{(1)}$ are polynomials of the first and $\psi_j^{(2)}$ polynomials of the second degree relative to $\xi_1, \ldots, \xi_{n-1}$. Some of the numbers of $\alpha_{nn}^{(j)}$ can equal zero. Let $\alpha_{nn}^{(1)} = 0$. Substituting the value $\xi_n = -\psi_1^{(2)}/\psi_1^{(1)}$ in the remaining $n - 1$ the equations, we obtain the system $n - 1$ of equations no higher than the fourth degree relative to $\xi_1, \ldots, \xi_{n-1}$

$$\Phi_j^{(n)}(\xi_1, \ldots, \xi_{n-1}) = 0 \qquad j = 1, \ldots, n-1 \qquad [3.16]$$

Where $\alpha_{nn}^{(j)} \neq 0$, consideration is as follows. Equations in [3.15] for some real $\xi_1, \ldots, \xi_{n-1}$ should have a real root $\xi_n = \xi_n^0$, i.e., the left-hand sides of these equations should be divided by $\xi - \xi_n^0$. Using the Euclidean algorithm (16) to find the overall largest divisor, for example for the first two equations in [3.15], we find that the last residue must equal zero. This residue conforms with the resultant and has the form

$$\chi_{12} \equiv (\alpha_{nn}^{(1)}\psi_2^{(2)} - \alpha_{nn}^{(2)}\psi_1^{(2)})^2 - (\alpha_{nn}^{(1)}\psi_2^{(1)} - \alpha_{nn}^{(2)}\psi_1^{(1)})\,(\psi_1^{(1)}\psi_2^{(2)} - \psi_2^{(1)}\psi_1^{(2)})$$

Equation $\chi_{12} = 0$ provides the first equation of the form [3.16].

In satisfying this condition the common root ξ_n^0 for the first two equations in [3.15] equal[19]

$$\xi_n^0 = \frac{\alpha_{nn}^{(1)}\psi_2^{(2)} - \alpha_{nn}^{(2)}\psi_1^{(2)}}{\alpha_{nn}^{(2)}\psi_1^{(1)} - \alpha_{nn}^{(1)}\psi_1^{(1)}} \qquad [3.17]$$

Substituting this value in the remaining $n - 2$ equations in [3.15] results in still other $n - 2$ equations of the fourth degree relative to $\xi_1, \ldots, \xi_{n-1}$. We return once more to equations of the form [3.16].

Let us deal in an analogous manner with these equations.

A still simpler equation results when some of $\alpha_{ij}^{(1)} = 0$.[20]

Let us assume that we obtained Equation [3.14]. The system of quadratic equations would have solutions for all β_j, sufficiently close to the assigned, if Equation [3.14] has even one real root for all γ_j, sufficiently close to the assigned. N is an even number; therefore the latter condition was equivalent to Equation [3.14] having only two different real roots. The question of the number of actual roots can be resolved by using the Sturm theorem (17), p. 291. Thus, the question of the existence of real solutions for the system of quadratic equations in [3.13], for any $n > 0$, can always be solved if some special cases are excluded by means of the finite number of rational algebraic operations.

Nevertheless, we are compelled to note that when $n \geq 4$ these algebraic computations are extremely awkward and their complexity grows very rapidly with increases of n.[21]

4′ We show that where vector a is a characteristic vector of matrix A

$$Aa = -\alpha a \qquad \alpha > 0 \qquad [3.18]$$

resolving Equation [2.18] permits an explicit solution.

Multiplying the relationship (III), [2.18], by a, we find

$$Ua = -(u, a) \times (A^* - \alpha I)^{-1} \qquad [3.19]$$

In addition, $\det(A^* - \alpha I) \neq 0$, since the spectrum of A lies in the left-hand semiplane.

The latter equations in [2.18] differ from the equations in [2.19] only by the fact that vector $b + c$ is used instead of vector b. Therefore, from formulas [3.1, 3.2 and 3.3], we have

$$\zeta \equiv (1/\sqrt{\rho})(u, A^{-1}a) = \sqrt{\rho} \pm \tilde{\Gamma}$$

where

$$\tilde{\Gamma}^2 = \rho + (b + c, A_a^{-1}) \qquad [3.20]$$

In our case, $\zeta = -[2\ (\alpha\sqrt{\rho})](u, a)$, hence

$$(u, a) = -\alpha\sqrt{\rho}[\sqrt{\rho} \pm \tilde{\Gamma}]$$

[18] Prof. D. K. Faddeyev pointed out this general algebraic result to the author. If some exceptions are regarded, this follows from (15), Part II, Section 83. The author is making use of this opportunity to thank Faddeyev.

[19] Actually, if the two quadratic equations $\alpha_0\xi^2 + \alpha_1\xi + \alpha_2 = 0$ and $\beta_0\xi^2 + \beta_1\xi + \beta_2 = 0$ have the common root $\xi = \xi_0$, then substituting the value $\xi = \xi_0$ in these equations and eliminating ξ_0^2 will yield $\xi_0 = (\beta_2\alpha_0 - \alpha_2\beta_0)/(\beta_0\alpha_1 - \beta_1\alpha_0)$.

[20] Evidently, in each concrete case generally a concrete form of the obtained equations should be used. In addition, a study should be made of cases where denominators in type [3.17] expression are converted to zero, as well as of those special cases when the last remainder in the Euclidean algorithm equals zero. It can be shown that these special conditions do not exist for $n < 5$. Beginning with $n = 5$ in the special conditions, instead of Equation [3.14], we obtained conditions of the form $P_m(\zeta) = 0$, $Q_s(\zeta) > 0$, i.e., some polynomial should have real roots, in which the other polynomial should be positive.

[21] With increases in n by unity the order of the final equation [3.14] grows two times. One can roughly estimate that with increases in n, algebraic computations grow by two times or more per unit change in n. In this connection, relationship [3.3] plays an important role, making possible a reduction in the number of quadratic equations per unit. In material which follows, we show that in the presence of the zero root for the characteristic equation of the open loop system, the number of quadratic equations per unit is still further lowered.

Substituting this value in [3.19] and using [2.18], (IV), we obtain

$$u = -\frac{1}{2}\left(A \pm \frac{\alpha\tilde{\Gamma}}{\sqrt{\rho}} I\right)^{-1} (A^* - \alpha I)(b + c) \quad [3.21]$$

An inverse matrix is known to exist if the minus sign is taken in expression [3.21]. The solution of u would be real, if $\tilde{\Gamma}$ were real, i.e., if the right-hand side of expression [3.20] were not negative.

Inasmuch as (see Eqs. [2.18], (I) and (II))

$$(c, A^{-1}a) = (A^{*-1}Ta, a) = (TA^{-1}a, a) = -\tfrac{1}{2}(A^{*-1}CA^{-1}a, a) < 0 \quad [3.22]$$

the only condition following from expression [3.20] which is necessary for the existence of a real solution[22] is

$$\Gamma^2 \equiv \rho + (b, A^{-1}a) > 0 \quad [3.23]$$

According to Point 2 of Theorem 2 the condition [3.23], i.e., condition $\Gamma^2 = \rho - (1/\alpha)(a, b) > 0$, is in our case a sufficient condition for overall stability.

Let us now consider the case where vector b is a characteristic vector of matrix A^*. We take the arbitrary solution x, σ in system [1.2]. We designate $\xi = (b, x)$. Using [1.2], we obtain

$$\begin{aligned} d\xi/dt &= -\beta\xi + (b, a)\varphi(\sigma) \\ d\sigma/dt &= \xi - \rho\varphi(\sigma) \end{aligned} \quad [3.24]$$

We obtained a system of such a form for $n = 1$, that relationship [3.18] was realized in a trivial form. According to formula [3.23] the overall stability of the system [3.24] has the form

$$\Gamma_1^2 \equiv \rho + 1 \times (-1/\beta) \times (b, a) > 0$$

It conforms with condition [3.23]. If this condition is satisfied $\xi \to 0$, $\sigma \to 0$ when $t \to \infty$. Then $\varphi(\sigma) \to 0$ when $t \to \infty$ and, from the first equation in [1.2] it is easy to find that $x \to 0$ when $t \to \infty$.

Let us combine the obtained results.

Suppose that in system [1.2] *the spectrum of matrix A lies in the left-hand semiplane and that either vector a is a characteristic vector of matrix A or vector b is a characteristic vector of matrix A*.*

For the overall stability of system [1.2] *for any function* $\varphi(\sigma)$, *satisfying conditions in* [1.4], *it is necessary and sufficient to satisfy the conditions in* [3.23].

5′ We saw, in the foregoing, that where $A^*b = -\beta b$, the system $n + 1$ of the nonlinear equations in [1.2] leads to the system of the two nonlinear equations of [3.24] and to one linear nonhomogeneous equation. Let us now consider some more general cases.

Assume, that in the sequence of vectors

$$v_1 = b \qquad v_2 = A^*b \qquad v_3 = A^{*2}b$$

the first p vectors are linearly independent, and $p + 1$ linearly dependent

$$v_{p+1} = \alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_p v_p$$

Evidently, $p \leq n$. We designate

$$\begin{aligned} \eta_1 &= b^*x \qquad \eta_2 = b^*Ax, \ \ldots, \ \eta_p = b^*Ax^{p-1} \\ \gamma_1 &= (a, b) \qquad \gamma_2 = (Aa, b), \ \ldots, \ \gamma_p = (A^{p-1}a, b) \end{aligned}$$

Multiplying the first equation in [1.2] from the left by b^*, b^*A, $\ldots$, b^*A^{p-1}, we obtain

$$\begin{aligned} \dot{\eta}_1 &= \eta_2 + \gamma_1\varphi(\sigma) \\ \dot{\eta}_2 &= \eta_3 + \gamma_2\varphi(\sigma) \\ &\ldots\ldots\ldots\ldots \\ \dot{\eta}_p &= \alpha_1\eta_1 + \ldots + \alpha_p\eta_p + \gamma_p\varphi(\sigma) \end{aligned}$$

Together with the second equation in [1.2], which can be written in the form

$$\dot{\sigma} = \eta_1 - \rho\varphi(\sigma)$$

these equations form a system of type [1.2], where

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \end{pmatrix} \qquad a = \begin{pmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \\ \gamma_4 \end{pmatrix} \qquad b = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

(as a demonstration take $p = 4$). Knowing $\sigma = \sigma(t)$ makes it possible to find $x = x(t)$ from the first equation of [1.2]. Thus, *if the minimum polynomial of vector a relative to matrix A has the degree* $p < n$, *the system* [1.2] *of an order* $n + 1$ *leads to an analogous system of an order* $p + 1$.

6′ We derive the stability conditions of system [1.2] for $n = 3.4$. First, let us transform system [1.2] to some special form. We designate

$$\begin{aligned} u_1 &= a \qquad u_2 = A^{-1}a \\ u_3 &= A^{-2}a, \ \ldots, \ u_n = A^{-n+1}a \end{aligned}$$

Assume that the determinant with columns $u_1, \ldots, u_n$ is nonzero

$$\delta \equiv \det\| a, A^{-1}a, A^{-2}a, \ldots, A^{-n+1}a \| \neq 0 \quad [3.25]$$

Let

$$-Au_1 = \alpha_1 u_1 + \alpha_2 u_2 + \ldots + \alpha_n u_n \quad [3.26]$$

$$x = \xi_1(t)u_1 + \ldots + \xi_n(t)u_n \quad [3.27]$$

Substituting [3.27] in the first equation of [1.2], we have

$$\begin{aligned} \dot{\xi}_1 &= -\alpha_1\xi_1 + \xi_2 + \varphi(\sigma) \\ \dot{\xi}_2 &= -\alpha_2\xi_1 + \xi_3 \\ &\ldots\ldots\ldots\ldots \\ \dot{\xi}_{n-1} &= -\alpha_{n-1}\xi_1 + \xi_n \\ \dot{\xi}_n &= -\alpha_n\xi_1 \end{aligned} \quad [3.28]$$

The second equation in [1.2] can be written in the form

$$\dot{\sigma} = \sum_{j=1}^{n} \beta_j\xi_j - \rho\varphi(\sigma) \quad [3.29]$$

where

$$\beta_1 = (a, b) \qquad \beta_2 = (A^{-1}a, b), \ \ldots, \ \beta_n = (A^{-n+1}a, b) \quad [3.30]$$

Equations in [3.28 and 3.29] form systems of the form [1.2], in which

$$A = \begin{pmatrix} -\alpha_1 & 1 & 0 & 0 & 0 \\ -\alpha_2 & 0 & 1 & 0 & 0 \\ -\alpha_3 & 0 & 0 & 1 & 0 \\ -\alpha_4 & 0 & 0 & 0 & 1 \\ -\alpha_5 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad a = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad b = \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \\ \beta_5 \end{pmatrix}$$

($n = 5$ is taken for definiteness). It is easy to see that

$$\det(A - \lambda I) = \lambda^n + \alpha_1\lambda^{n-1} + \ldots + \alpha_n \quad [3.31]$$

Inasmuch as $\det(A - \lambda I)$ does not depend on the selection of a basis, one may consider that in relationship [3.31] A is the originally assigned matrix.

We will consider that the roots of the equation

$$\det(A - \lambda I) \equiv \lambda^n + \alpha_1\lambda^{n-1} + \ldots + \alpha_n = 0$$

[22] In deriving the inequalities in [3.22 and 3.23], the [3.18] relationships were not used. Therefore, the inequality [3.23] is also necessary for the existence of a real solution u in Equations [2.18], (III) and (IV).

lie in the left semiplane. Then

$$\alpha_1 > 0, \ldots, \alpha_n > 0$$

Let us consider Equation [2.19]. We designate

$$u = \begin{pmatrix} \zeta_1 \\ \zeta_2 \\ \cdot \\ \cdot \\ \zeta_n \end{pmatrix} \qquad U = \begin{pmatrix} [1,1], & \ldots, & [1,n] \\ \cdots\cdots & & \\ \cdots\cdots & & \\ \cdots\cdots & & \\ [n,1], & \ldots, & [n,n] \end{pmatrix} \qquad [ij] = [ji]$$

First we consider the case $n = 3$. It is easy to find the value ζ_2 from equations in [2.19], written in a scalar form; then, expressing ζ_1 through ζ_3, we obtain, for $\zeta = \zeta_3$ the equation[23]

$$\zeta^4 + 4\delta_0\xi^3 + 4(\delta_0^2 + \chi)\zeta^2 + 8\delta_0\left(\chi + \frac{\alpha_3\rho\delta_0}{\alpha_2^2}\right)\zeta + 4\left(\chi + \frac{\delta_0^2\delta_1}{\alpha_2^2}\right) = 0 \quad [3.32]$$

where

$$\Gamma^2 = \rho + \beta_2$$
$$\delta_0 = (\alpha_2/\alpha_3)(\pm\sqrt{\rho}\Gamma)$$
$$\chi = (\rho/2\alpha_3)(\alpha_1\beta_2 - \alpha_2\beta_3 + 2\alpha_1\rho - \beta_1)$$
$$\delta_1 = \rho\sum_{j=1}^{3}\alpha_j\beta_j + 2\alpha_2\rho^2 + 2\alpha_3\rho\delta_0 - \alpha_1^2\rho^2$$
$$\cdots\cdots\cdots\cdots [3.33]$$

Using Theorem 2, Point 1, we obtain the following result.

System [1.2] *was given for* $n = 3$. *Assume that the "characteristic equation of the open loop system"*

$$\det(A - \lambda I) \equiv \lambda^3 + \alpha_1\lambda^2 + \alpha_2\lambda + \alpha_3 = 0$$

has roots in the left semiplane (i.e., $\alpha_1 > 0$, $\alpha_2 > 0$, $\alpha_1\alpha_2 > \alpha_3 > 0$) *and that inequality* [3.25] *is realized. We determine the numbers of* β_j *according to the formulas in* [3.30] *and the numbers in* [3.33]. *If Equation* [3.32] *has at least two real roots, then system* [1.2] *is stable on the whole.*

In order to express stability conditions directly through the coefficients of the assigned system we first prove the following simple algebraic lemma.

LEMMA 3 *Assume that the equation*

$$\psi(\zeta) = \zeta^m + \delta_1\zeta^{m-1} + \ldots + \delta_m = 0$$

has no multiple roots. Let

$$\psi_0(\zeta) = \psi(\zeta)$$
$$\psi_1(\zeta) = m\zeta^{m-1} + \ldots$$
$$\psi_2(\zeta) = \epsilon_2\zeta^{m-2} + \ldots \qquad \epsilon_2 \neq 0$$
$$\cdots\cdots\cdots\cdots$$
$$\psi_m(\zeta) = \epsilon_m \qquad \epsilon_m \neq 0$$

be the sequence of the Sturm function for equation $\psi(\zeta) = 0$. *Then, the number of changes of signs* k *in the series of the highest order coefficients in the Sturm function*

$$\epsilon_0 = 1, \epsilon_1 = m, \epsilon_2, \epsilon_3, \ldots, \epsilon_m \quad [3.34]$$

equals the number of pairs of complex roots in the equation.

Proof. Designate d as the number of actual roots; we have to prove formula

$$d + 2k = m \quad [3.35]$$

In conformance with the Sturm rule (17), the number d equals the increase in the number of sign changes in the series of Sturm functions for $x = -\infty$ as compared with $x = +\infty$. Thus, designating k' as the number of sign changes in the series

$$(-1)^m, \ldots, (-1)^{i+m}\epsilon_i, \ldots, \epsilon_{m-2}, -\epsilon_{m-1}, \epsilon_m \quad [3.36]$$

we obtain $d = k' - k$. Therefore, we must prove that

$$k + k' = m \quad [3.37]$$

Let us designate $K[\epsilon, \epsilon', \epsilon'', \ldots]$ as the number of sign changes to series $\epsilon, \epsilon', \epsilon'', \ldots$. We have

$$k + k' = K[\epsilon_0, \epsilon_1] + \ldots + K[\epsilon_1, \epsilon_2] + \ldots + K[(-1)^m\epsilon_0, (-1)^{m-1}\epsilon_1] + K[(-1)^{m-1}\epsilon_1, (-1)^m\epsilon_2] + \ldots \quad [3.38]$$

It is easy to prove that always

$$K[\epsilon, \epsilon'] + K[\epsilon, -\epsilon'] = 1$$

Therefore, adding [3.38] in columns, we obtain [3.37]. The lemma is proved.

For equations of the fourth degree, the condition $d \geq 2$ denotes that $k = 0$ or 1. In this case, only the following combination of signs is possible

ϵ_0	ϵ_1	ϵ_2	ϵ_3	ϵ_4	k
+	+	+	+	+	0
+	+	+	+	−	1
+	+	+	−	−	1
+	+	−	−	−	1

Therefore, condition $d \geq 2$ denotes that either $\epsilon_2 > 0$, $\epsilon_3 > 0$ or $\epsilon_3 < 0$, $\epsilon_4 < 0$. Determining ϵ_2, ϵ_3, ϵ_4 from the formulas in (6), p. 226, we obtain the following result.

System [1.2] *was given for* $n = 3$ *with the previous assumptions. If the inequalities*

$$\alpha_2(\alpha_2\beta_2 + \alpha_3\beta_3 + \alpha_2\rho) > \alpha_3[\alpha_1(\beta_2 + \rho) + (\alpha_1\rho - \beta_1)]$$
$$[\alpha_2(\alpha_2\beta_2 + \alpha_3\beta_3 + \alpha_2\rho) - \alpha_1\alpha_3(\rho + \beta_2) - \alpha_3(\alpha_1\rho - \beta_1)] \times \left[\alpha_1^2\rho - 2\rho\alpha_2 - \sum_{j=1}^{3}\alpha_j\beta_j\right] > 9\alpha_3^2\rho(\rho + \beta_2)$$
$$\cdots\cdots\cdots\cdots [3.39]$$

or the inequality

$$[\alpha_2(\alpha_2\beta_2 + \alpha_3\beta_3 + \alpha_2\rho) - \alpha_1\alpha_3(\rho + \beta_2) - \alpha_3(\alpha_1\rho - \beta_1)] \times \left[\alpha_1^2\rho - 2\alpha_2\rho - \sum_{j=1}^{3}\alpha_j\beta_j\right] < 9\alpha_3^2\rho(\rho + \beta_2) < 0$$
$$A_2 \neq 0 \qquad A_3 \neq 0$$
$$\frac{B_3}{A_3}\left(B_2 - \frac{A_2B_3}{A_3}\right) < C_2$$
$$\cdots\cdots\cdots\cdots [3.40]$$

are satisfied where

$$A_2 = \delta_0^2 - 2\chi$$
$$B_2 = 2\delta_0(\delta_0^2 - 2\chi) - \frac{6\alpha_3\rho}{\alpha_2^2}\delta_0^2$$
$$C_2 = 2(\delta_0^2 - 2\chi)\chi + 2\frac{\delta_0^2}{\alpha_2^2}(\alpha_3\rho\delta_0 - 2\delta_1)$$
$$A_3 = \frac{16\delta_0^2}{\alpha_2^2(\delta_0^2 - 2\chi)^2}\left[(\delta_0^2 - 2\chi)(2\alpha_3\delta_0\rho - \delta_1) - 9\frac{\alpha_3^2\rho^2\delta_0^2}{\alpha_2^2}\right]$$
$$B_3 = \frac{16\delta_0^2}{\alpha_2^2(\delta_0^2 - 2\chi)^2}\left[(\delta_0^2 - 2\chi)(4\alpha_3\rho\chi - \delta_1\delta_0) + 3\frac{\alpha_3\rho\delta_0^2}{\alpha_2^2}(\alpha_3\rho\delta_0 - 2\delta_1)\right]$$

then system [1.2] *is stable on the whole.*

Here again α_j are coefficients of the "open loop system" of equations; the remaining numbers are determined by formulas [3.30 and 3.33].

Suppose that in system [2.1] *for* $n = 4$, *the "characteristic*

[23] Corresponding computations were made in a thesis by A. B. Konstantinova. (In particular, she obtained expressions [3.32, 3.33 and 3.39 − 3.42]). The author is using this opportunity to thank A. B. Konstantinova for her work.

equation of the open loop system"

$$\det (A - \lambda I) \equiv \lambda^4 + \alpha_1\lambda^3 + \alpha_2\lambda^2 + \alpha_3\lambda + \alpha_4 = 0$$

has roots in the left-hand semiplane and the inequality [3.25] *is realized.*

In order to obtain sufficient stability conditions we proceed in the following manner.

1 *We determine the numbers* β_j, *using the formulas in* [3.30], *and the coefficients*

$$\Gamma^2 = \rho + \beta_2$$

$$R = \frac{\alpha_4}{2(\pm\sqrt{\rho}\Gamma)}$$

$$P = \alpha_2\alpha_3\alpha_4 + 2\alpha_3\beta_4R^2$$

$$Q = -\frac{\rho}{2}\alpha_3\beta_1 + \alpha_1\alpha_3\rho^2 + \alpha_1\alpha_3\beta_2\frac{\rho}{2} - \alpha_4\rho^2 - \frac{\alpha_4^2\rho}{2R} - \frac{\alpha_4\beta_2\rho}{2} + \frac{R^2\beta_4}{2} + \frac{\alpha_1\alpha_4\beta_3\rho}{2} - \frac{\alpha_2\alpha_3\beta_3\rho}{2}$$

$$T = \tfrac{1}{2}\alpha_3^2 + \tfrac{1}{2}\alpha_2\alpha_4 + \rho\beta_4R^2$$

$$S = \rho_3^2\beta_4R + \alpha_2\rho^2 + \frac{\alpha_2\alpha_4\rho}{2R} + \frac{\rho}{2}\sum_{j=1}^{4}\alpha_j\beta_j - \frac{\alpha_1^2\rho^2}{2}$$

. . . . . [3.41]

2 *Let us construct the equation*

$$R^4\zeta^8 + 4\alpha_3R^3\zeta^7 + 4R^2(\alpha_3^2 + T)\zeta^6 + 2R(4\alpha_3T + P)\zeta^5 + 4(T^2 + \alpha_3P + R^2Q + 2\alpha_4^2R\rho)\zeta^4 + 4\left(2\alpha_3RQ + \frac{TP}{R} + 4\alpha_3\alpha_4^2\rho\right)\zeta^3 + \left[\frac{P^2}{R^2} + 8TQ + \alpha_4^2\left(10\alpha_3^2\rho\frac{1}{R} + 8S\right)\right]\zeta^2 + 2\left[2\frac{PQ}{R} + \frac{\alpha_3\alpha_4^2}{R}\left(\frac{\alpha_3}{R} + 4S\right)\right]\zeta + 4Q^2 + 2\alpha_3^2\alpha_4^2 \times \frac{S}{R^2} = 0$$

[3.42]

3 *If* $\zeta = -\alpha_3/2R$ *is not a root of Equation* [3.42] *and if Equation* [3.42] *has at least two real roots which are different, then system* [1.2] *is stable on the whole.*

If $\zeta = -\alpha^3/2R$ *is a root of Equation* [3.42] *and if the inequality*

$$\alpha_1^2\rho^2 - 2\sum_{j=1}^{4}\alpha_j\beta_j - \frac{\alpha_2\alpha_4\rho}{R} - 2\alpha_2\rho^2 - 2\rho R\left(\frac{\alpha_3^2}{4R^2} + \rho\beta_4\right) + \frac{\alpha_3^2\rho}{R} > 0$$

is satisfied then system [1.2] *is stable on the whole.*

Equation [3.39] was obtained following elimination of ζ_1, ζ_2, ζ_4 and $\zeta = \zeta_3$ in the system of resolving equations.

We note that the formulated conditions of stability were expressed through numbers α_j, β_j, ρ, which are, as can be easily demonstrated, invariants of the linear transformation

$$x = Fx_1$$

of the basic variables in system [1.2].

Finally, we note that if in the assigned system [1.2], the number of parameters is small, it is more convenient to use not the formulas given above but Theorem 2 directly, forming and studying, in each concrete case, the resolving equations.

Section 4 Case of Zero Roots of the Characteristic Equation in the Open Loop System

Let us now consider the case where $\det A = 0$.

If the rank of matrix A is less than $n - 1$, then in formula [3.4], $Q = 0$; hence, it follows that the determinant of matrix [1.5] identically equals zero, i.e., the necessary overall stability conditions are not satisfied (even for a linear system). We therefore propose that the rank of matrix A equal $n - 1$.

We designate k the multiplicity of the characteristic value $\lambda = 0$. Two cases are possible: $k = 1$ and $k > 1$. In the latter case, one box in the canonical form of the Jordan matrix A corresponds to the proper value $\lambda = 0$. (In other words, the rank of matrix A would be less than $n - 1$.)

1′ Suppose that $k = 1$. Let y_0, z_0 be any vectors, satisfying equations

$$Ay_0 = 0 \qquad A^*z_0 = 0 \qquad [4.1]$$

and S some real, nonsingular matrix, in which the first $n - 1$ columns of s_j were orthogonal to vector z_0, and the latter equaled y_0. We show that then

$$S^{-1}AS = \begin{pmatrix} A' & 0 \\ 0 & 0 \end{pmatrix} \qquad [4.2]$$

where A' is a nonsingular matrix. We designate

$\mathbf{R}$ = n-dimensional space
$\mathbf{D}$ = subspace of values of operator A
$\mathbf{D}^*$ = same for operator A^*
$\mathbf{N}$ = $\{\lambda y_0\}$
$\mathbf{N}^*$ = $\{\lambda z_0\}$

From general theorems of linear algebra it follows[24]

$$\mathbf{R} = \mathbf{N} + \mathbf{D} \qquad \mathbf{R} = \mathbf{N}\oplus\mathbf{D}^* \qquad \mathbf{R} = \mathbf{N}^*\oplus\mathbf{D} \qquad [4.3]$$

which, however, is easy to prove even directly.

Thus, condition $(s_j, z_0) = 0$ denotes that $s_j \in \mathbf{D}$. Since $\mathbf{D}$ is invariant relative to A, we have

$$As_i = \sum_{j=1}^{n-1} a_{ij}s_j \qquad Ay_0 = 0$$

These n equalities can be written in the form

$$AS = S\begin{pmatrix} A' & 0 \\ 0 & 0 \end{pmatrix} \qquad A' = \|\alpha_{ij}\|_{ij=1}^{n-1}$$

from which [4.2] also follows.

We designate

$$S^{-1}x = \begin{pmatrix} x' \\ \xi \end{pmatrix} \qquad S^{-1}a = \begin{pmatrix} a' \\ \alpha' \end{pmatrix} \qquad S^*b = \begin{pmatrix} b' \\ \beta \end{pmatrix}$$

Multiplying the first equation in [1.2] from the left by S^{-1}, we write system [1.2] in the form

$$dx'/dt = A'x' + a'\varphi(\sigma)$$
$$d\xi/dt = \alpha\varphi(\sigma)$$
$$d\sigma/dt = (b', x') + \beta\xi - \rho\varphi(\sigma)$$

. . . . . [4.4]

In substituting $\varphi(\sigma) = \mu\sigma$ we obtain a linear system with matrix

$$A_\mu^{(0)} = \begin{Vmatrix} A' & 0 & \mu a' \\ 0 & 0 & \mu\alpha \\ b'^* & \beta & -\mu\rho \end{Vmatrix}$$

Just as before, we derive the necessary condition of stability. We use formula [3.4]. For this we note that in our case the matrix Q has the form

$$Q = \begin{Vmatrix} 0 & 0 \\ 0 & \Delta' \end{Vmatrix} \qquad \Delta' = \det A'$$

Therefore, formula [3.4] provides

$$\det A_\mu^{(0)} = -\mu\Delta'\alpha\beta$$

[24] Sign + denotes expansion into a straight sum of the nonintersecting subspaces, sign $\oplus$ denotes that the corresponding subspaces are orthogonal. The second and third formulas in [4.3] were correct without any assumption concerning the character of the zero characteristic values of A. ($\mathbf{N}$ and $\mathbf{N}^*$ then denote the characteristic subspaces of operators A and A^* for $\lambda = 0$.) The first equality in [4.2] denotes that simple, elementary divisors correspond to the characteristic value $\lambda = 0$.

In case of stability, when $\mu > 0$, we have sign det $A_\mu^{(0)} = (-1)^{n+1}$, sign $\Delta' = (-1)^{n-1}$. Therefore, a necessary condition for stability is

$$\alpha\beta < 0 \qquad [4.5]$$

Let us now consider the equations in [2.19][25]

$$S^*US = \begin{pmatrix} U' & v \\ v^* & \psi \end{pmatrix} \qquad S^*u = \begin{pmatrix} u' \\ \eta \end{pmatrix}$$

Multiplying the first equation in [2.19] from the left by S^* and from the right by S, and the second from the left by S^*, we find that $\eta = 0$, $v = 0$, while the remaining equations take on the form

$$\begin{gathered} A'^*U' + U'A = -u'u'^* \\ U'a' + \rho u' + \tfrac{1}{2}\rho b' = 0 \\ \psi\alpha + \tfrac{1}{2}\rho\beta = 0 \end{gathered} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots [4.6]$$

The last equation can be discarded (this equation can always be satisfied when fulfilling the conditions in [4.5]). The first two equations provide a system of quadratic equations of the very same form as in [2.19], but now det $A' \neq 0$. The necessary condition in [3.2] for the existence of a real solution is written in the form[26]

$$\Gamma^2 = \rho + (b', A'^{-1}a') > 0 \qquad [4.7]$$

and instead of [3.1 and 3.3], we obtain

$$\xi \equiv \frac{1}{\sqrt{\rho}}(u', A'^{-1}a') = \sqrt{\rho} \pm \Gamma \qquad [4.8]$$

The necessary conditions in [4.5 and 4.7] and relationship [4.8] have, however, the shortcoming that they are written in a special system of coordinates. We now obtain analogous invariant relationships.

Multiplying the first equation in [2.19] from the left by x_0^* and from the right by x_0, and the second equation in [2.19] from the left by x_0^*, where x_0 is still an arbitrary vector, we obtain

$$\begin{gathered} 2(Ax_0, Ux_0) = -(u, x_0)^2 \\ (Ua, x_0) + \rho(u, x_0) + \tfrac{1}{2}\rho(b, x_0) = 0 \end{gathered} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots [4.9]$$

We designate $\xi = (1/\sqrt{\rho})(u, x_0)$ and require that

$$(Ax_0, Ux_0) = (Ua, x_0) \qquad [4.10]$$

Then, it will follow, from the equations in [4.9], that

$$\zeta^2 - 2\sqrt{\rho}\zeta - (b, x_0) = 0$$

Hence, just as before, we obtain

$$\Gamma^2 \equiv \rho + (b, x_0) > 0 \qquad [4.11]$$

$$\zeta \equiv (1/\sqrt{\rho})(u, x_0) = \sqrt{\rho} \pm \Gamma \qquad [4.12]$$

It remained for us to determine vector x_0, satisfying Equation [4.10]. Let us consider the equation

$$Ax = a - y \qquad (y, Ux) = 0 \qquad [4.13]$$

If $x = x_0$, $y = y_0$ are any solutions to these equations, then the corresponding vector satisfies Equation [4.10]. (Evidently, the reverse is also correct.)

We show that with the additional conditions

$$x \in \mathbf{D} \qquad y \in \mathbf{N} \qquad [4.14]$$

vectors x and y, satisfying the first equation in [4.13], are determined identically and satisfy the second equation in [4.13].

The existence and uniqueness follow directly from the first relationship in [4.2]. Let us show the correctness of the second relationship [4.13].

In the equations of [4.9], x_0 is any vector. Setting $x_0 = y \in \mathbf{N}$, we obtain $(u, y) = 0$, i.e., $u \in \mathbf{D}^*$. Multiplying the first equation in [2.19] from the left by y, we obtain

$$A^*Uy + UAy = -u(u, y) = 0$$

and since $Ay = 0$, then $A^*Uy = 0$, $Uy \in \mathbf{N}^*$. From the third expansion in [4.2] we have $(Uy, x) = 0$; hence the second relationship [4.13] also follows.

Vectors x_0, y_0 can be determined in the following way. In conformance with the first and third expansion in [4.3], we have

$$a = a_1 + y_0 \qquad Ay_0 = 0 \qquad (a_1, z_0) = 0 \qquad [4.15]$$

Vectors a_1 and y_0 are uniquely determined. (By the same token, vector y_0, determined by relationship [4.1] to within a factor, was now fully determined.) Vector x_0 with relationships

$$Ax_0 = a_1 \qquad (x_0, z_0) = 0 \qquad [4.16]$$

is also determined identically. Vectors $x = x_0$, $y = y_0$ satisfy the first equation in [4.13] and the relationships in [4.14], and consequently the second equation in [4.13] as well.

Multiplying the vector equalities in [4.15 and 4.16] from the left by S^{-1} and using the uniqueness of solutions x_0, y_0 in any coordinate system, we obtain

$$Sa_1^{-1} = \begin{pmatrix} a' \\ 0 \end{pmatrix} \qquad Sx_0^{-1} = \begin{pmatrix} A'^{-1} & a' \\ & 0 \end{pmatrix}$$

$$S^{-1}y_0 = \begin{pmatrix} 0 \\ a \end{pmatrix} \qquad Sz_0^* = \text{const} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Thus, the necessary conditions for [4.7 and 4.11] coincide: They are written only in different coordinate systems. (It is easy to see that Γ^2 is invariant when transforming $x = Sx_1$ of system [1.2].) Relationship [4.8] coincides with [4.12] in exactly the same way.

Let us now write the condition [4.5] in an invariant form. From an expansion of $\mathbf{R} = \mathbf{N}^* + \mathbf{D}^*$, analogous to the first expansion in [4.3], and from the second expansion in [4.3], it follows that

$$b = b_1 + z_0 \qquad A^*z_0 = 0 \qquad (b_1, y_0) = 0 \qquad [4.17]$$

Vectors b_1 and z_0 are determined identically. (By the same token, vector z_0, determined earlier to within a factor, is now also fully determined.) Multiplying the first equation in [4.17] by S^* and employing the uniqueness of the expansion in [4.17], we obtain

$$S^*b_1 = \begin{pmatrix} b' \\ 0 \end{pmatrix} \qquad S^*z_0 = \begin{pmatrix} 0 \\ \beta \end{pmatrix}$$

Hence $\alpha\beta = (y_0, z_0)$, and condition [4.5] is written in the form

$$(y_0, z_0) < 0 \qquad [4.18]$$

Finally, vectors x_0, y_0, z_0 are determined identically with formulas in [4.15, 4.16 and 4.17]. The necessary conditions have the form [4.11 and 4.18]. Besides, $(u, y_0) = 0$ and relationship [4.12] takes place, i.e., with the presence of one zero root the number of quadratic equations in the resolving system of equations is reduced by a unit.

2′ Let us now consider the case $k \geq 1$. Where $k > 1$, as was indicated in the foregoing, it is possible to be restricted to the case when one canonical box of an order of k corresponds to the characteristic value $\lambda = 0$. The nonsingular

[25] Note that we still cannot maintain that the existence of a real solution for these equations provides overall stability. This will be proved in Section 5, Theorem 6.

[26] Now, however, we still don't know whether this condition is necessary even for stability. In material which follows, we show that it is necessary that $\Gamma^2 \geq 0$ for stability.

matrix S exists in such a way that

$$S^{-1}AS = \left\| \begin{array}{c|ccc} A' & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \\ & 0 & 0 & 1 \\ & 0 & 0 & 0 \end{array} \right\| \equiv A_1 \qquad [4.19]$$

(For simplicity here we took $k = 3$.) We designate

$$S^{-1}a = \begin{pmatrix} a' \\ a'' \end{pmatrix} \equiv a_1 \qquad S^*b = \begin{pmatrix} b' \\ b'' \end{pmatrix} \equiv b_1$$

where one prime refers to vectors of an order of $n - k$, and two primes to vectors of an order of k. Multiplying the first equation in [1.2] from the left by S^{-1} and letting $\varphi(\sigma) = \mu\sigma$, we obtain a linear system with matrix

$$A_\mu^{(0)} = \left\| \begin{array}{cc} A_1 & \mu a_1 \\ b_1^* & -\mu\rho \end{array} \right\| \qquad [4.20]$$

According to formula [3.4], the characteristic equation has the form

$$\det (A_\mu^{(0)} - \lambda I) \equiv -(\mu\rho + \lambda) \times \Delta(\lambda) - \mu[Q_1(\lambda)a_1, b_1] = 0 \qquad [4.21]$$

where $\Delta(\lambda) = \det (A_1 - \lambda I)$, $Q_1(\lambda)$ is the transposed matrix of algebraic supplements to matrix $A_1 - \lambda I$. From formula [4.19], we have $\Delta(\lambda) = (-1)^k\lambda^k \times \Delta'(\lambda)$, $\Delta'(\lambda) = \det (A' - \lambda I)$. Further, $Q_1(\lambda)$ is equal to $\Delta(\lambda) \times (A_1 - \lambda I)^{-1}$; therefore

$$Q_1(\lambda) = -\Delta(\lambda) \times \left\| \begin{array}{c|ccc} -(A' - \lambda I)^{-1} & 0 & 0 & 0 \\ \hline 0 & \lambda^{-1} & \lambda^{-2} & \lambda^{-3} \\ & 0 & \lambda^{-1} & \lambda^{-2} \\ & 0 & 0 & \lambda^{-1} \end{array} \right\| =$$

$$(-1)^{k-1} \times \left\| \begin{array}{c|cc} -\lambda^k Q'(\lambda) & 0 & 0 \\ \hline 0 & \Delta'(\lambda) \times R(\lambda) & \end{array} \right\| \qquad [4.22]$$

where $Q'(\lambda)$ is the transposed matrix of algebraic supplements to matrix $A' - \lambda I$ and

$$R(\lambda) = \begin{vmatrix} \lambda^2 & \lambda & 1 \\ 0 & \lambda^2 & \lambda \\ 0 & 0 & \lambda^2 \end{vmatrix}$$

(In the general case λ^{k-1} stands on the main diagonal.) Equation [4.21] assumes the form

$$(-1)^{k-1} \det (A_\mu^{(0)} - \lambda I) \equiv \mu\psi(\lambda) + \lambda^{k+1}\Delta'(\lambda) \equiv 0 \qquad [4.23]$$

where

$$\psi(\lambda) = \rho\lambda^k\Delta'(\lambda) + \lambda^k[Q'(\lambda)a', b'] - [R(\lambda)a'', b''] \times \Delta'(\lambda) \qquad [4.24]$$

Hence, where $\lambda = 0$

$$(-1)^{k-1} \det A_\mu^{(0)} = -\mu \times \Delta'(0)\alpha\beta$$

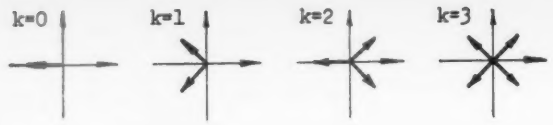
where

$$a'' = \begin{pmatrix} \cdot \\ \cdot \\ \cdot \\ \alpha \end{pmatrix} \qquad b'' = \begin{pmatrix} \beta \\ \cdot \\ \cdot \\ \cdot \end{pmatrix}$$

Where there is stability sign $\det A_\mu^{(0)} = (-1)^{n+1}$, sign $\Delta'(0) = (-1)^{n-k}$; thus, the necessary condition of stability again has the form of [4.5].

When $\mu = 0$ Equation [4.23] has $k + 1$ roots $\lambda = 0$. Using the basic theorems on the algebraic functions (14), Part 1, we find that these $k + 1$ roots are determined when μ is sufficiently small by formulas

$$\tau = (\alpha\beta\mu)^{1/(k+1)}$$
$$\lambda = \tau + [\gamma/(k+1)]\tau^2 + \ldots$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots [4.25]$$



where

$$\gamma = \frac{\partial}{\partial\lambda}\left[\ln \frac{\psi(\lambda)}{\Delta'(\lambda)}\right]_{\lambda=0}$$

According to the condition $\alpha\beta\mu < 0$. Therefore, when $k > 1$ at least one of the values of τ will have a positive real part. Consequently, when $k > 1$ and μ is sufficiently small, the matrix $A_\mu^{(0)}$ would have characteristic values $\lambda_j^{(0)}$ with $\mathrm{Re}\lambda_j^{(0)} > 0$. The illustration shows the nature of the behavior, with a sufficiently small $\mu > 0$, of characteristic values of matrix $A_\mu^{(0)}$, becoming zero when $\mu = 0$ (for $k = 0$ the formula [3.6], and for $k = 1, 2, 3$ the formula [4.25]).

When $k = 1$, Im $\tau = 0$, and from [4.25], the condition of stability has the form $\gamma \geq 0$. When $k = 1$

$$\psi(\lambda) = \rho\lambda\Delta'(\lambda) + \lambda[Q'(\lambda)a', b'] - \Delta'(\lambda)\alpha\beta$$

Hence

$$\psi(\lambda)/\Delta'(\lambda) = \lambda\rho + \lambda([A' - \lambda I]^{-1}a', b') - \alpha\beta$$

and

$$\gamma = \frac{\rho + (A'^{-1}a', b')}{-\alpha\beta} = -\frac{\Gamma^2}{(y_0, z_0)}$$

(See Eq. [4.7].) Thus, when $k = 1$ the expansion of [4.25] has the form

$$\tau_{1,2} = \pm i\sqrt{-(y_0, z_0)\mu}$$
$$\lambda_{1,2} = \tau_{1,2} - \frac{\Gamma^2}{(y_0, z_0)}\tau_{1,2}^2 + \ldots$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots [4.26]$$

Thus, it turned out that condition $\Gamma^2 \geq 0$ is not only necessary for the system of resolving equations to have real solutions, but necessary as well for stability.

Let us combine the basic results of Section 4.

1 *Assume that in system* [1.2] *the matrix A has one zero characteristic value and that the remainder have negative real parts. Vectors* x_0, y_0, z_0 *are determined uniquely by the relationships* [4.15, 4.16 *and* 4.17]. *The necessary conditions of stability*[27] *have the form*

$$(y_0, z_0) < 0 \qquad \Gamma^2 = \rho + (b, x_0) \geq 0$$

Expansion of the two characteristic values of the linearized system [$\varphi(\sigma) = \mu\sigma$], *becomes zero when* $\mu = 0$ *has the form* [4.26].

2 *If in system* [1.2] *matrix A has a zero characteristic value of multiplicity* $k > 1$, *then even the linear system* [1.2] *with* $\varphi(\sigma) = \mu\sigma$ *and with a sufficiently small* μ, *has unlimited solutions when* $t \to \infty$.

Section 5 Overall Stability in the Presence of a Zero Root of an Open Loop System[28]

We will again seek a Lyapunov function of the form [2.1], where $H > 0$, for system [1.2]. In our case, however, $\dot{V}$ cannot be determined as a negative function. Actually, the

[27] Even for the linear system $\varphi(\sigma) = \mu\sigma$.

[28] The results of this paragraph especially provide a rigorous basis to the statement in (1), p. 71. We are reminded that in addition, unlike (1), we are considering the Lyapunov function in form [2.1] where H is the arbitrary positively definite matrix.

expression for $\dot{V}$ can be reduced to the form (see Eqs. [2.2, 2.3, 2.4 and 2.5])

$$\dot{V} = -\frac{\rho}{2}(Cx, x) - \frac{1}{\rho}(\rho\varphi + g^*x)^2 \qquad [5.1]$$

If $\dot{V} < 0$ then we would have $C > 0$; hence (see Eq. [2.5]) $A^*H + HA < 0$ and according to the lemma in Section 2 it would have resulted that matrix A has all characteristic values with negative real parts. Thus, it is only possible to assume that $\dot{V} \leqslant 0$.

We show that in transition to system [4.4], the Lyapunov function assumes the form

$$V = (H'x', x') + \chi\xi^2 + \int_0^\sigma \varphi(\sigma)d\sigma \qquad [5.2]$$

(there will be an absence of product ξ and component of vector x'). From Equations [2.3 and 4.1] we have for the arbitrary vector x

$$(Gy_0, x) = -(A^*Hy_0, x)$$

Letting $x = y_0$, we obtain $(Gy_0, y_0) = 0$ and since $G > (1/\rho)gg^* \geq 0$, then

$$Gy_0 = 0 \qquad (g, y_0) = 0 \qquad [5.3]$$

Hence $(A^*Hy_0, x) = 0$ for any x. Consequently $A^*Hy_0 \doteq 0$

$$Hy_0 = \lambda z_0 \qquad [5.4]$$

In transition to system [4.4], matrix H is replaced by matrix

$$S^*HS = \begin{pmatrix} H' & h' \\ h'^* & \chi \end{pmatrix}$$

Elements of vector h' have the form $s_j^*Hy_0 = (Hy_0, s_j) = \lambda(z_0, s_j) = 0$ according to the condition of selecting vectors s_j. Thus $h' = 0$ and the equality [5.2] was proved. Simultaneously, we obtained the relationships in [5.3], from which follows (see Eqs. [2.3 and 2.5])

$$Cy_0 = 0 \qquad [5.5]$$

Taking the product from the function in [5.2], by virtue of system [4.4], we obtain[29]

$$\dot{V} = -(C'x', x') - (1/\rho)[\rho\varphi + (g', x')]^2 + (2\alpha\chi + \beta)\xi\varphi$$

where

$$\begin{aligned} -g' &= H'a' + \tfrac{1}{2}b' \\ -C' &= A'^*H' + H'A' - (1/\rho)g'g'^* \end{aligned} \qquad [5.6]$$

If $2\alpha\chi + \beta \neq 0$, then V is not a constant function. Therefore

$$2\alpha\chi + \beta = 0 \qquad [5.7]$$

and

$$\dot{V} = -(C'x', x') - (1/\rho)[\rho\varphi + (g', x')]^2 \qquad [5.8]$$

If $\dot{V} \leq 0$, then $C' \geq 0$. We will assume that

$$C' > 0 \qquad [5.9]$$

This conforms to the minimum possible degeneration of function $\dot{V}$. In the original basis this condition has the form

$$(Cx, x) > 0 \text{ when } x \neq \text{const} \times y \qquad [5.10]$$

The first two equations in [4.6] have the same form as the equations in [2.19]. Therefore, the presence of a real solution u' (for the assigned vector b' and vectors b', as close as desired to the assigned) denotes the presence in system [4.4] of a positively definite Lyapunov function of the form [5.2], whose derivative is not positive and has the form [5.1], where C' is a sufficiently small positively definite matrix.

In the original basis, the presence of a real solution u in equations of [2.19] denotes the existence, for system [1.2], of a positively definite Lyapunov function form [2.1]; its derivative is not positive, has the form [5.1] and is minimally degenerated. Matrix C satisfies the relationships in [5.5 and 5.10].

In addition, however, as we have seen, matrix U from the first equation of [2.19] is determined differently. Actually, element ψ in the equations of [4.6] is not determined by the first equation in [2.19]; it is determined by the second equation. Thus, this dissimilarity has no influence on the one-valued form of the resolving equations. If the Lyapunov function has no interest to us by itself, then the equation for determining ψ can simply be discarded.

We will prove overall stability, referring to the theorems of Barbashin and Krasovskii (8).

For this, we must show that there are no positive semi-trajectories in system [1.2] on sets of $V \equiv 0$. This is evident. If $V \equiv 0$, then from [5.8] we have $x' \equiv 0$, $\sigma \equiv 0$. Then $\xi \equiv 0$ follows from system [4.4]. Finally, for reference to Theorem 4 (8) it is necessary to demand the existence of the continuous derivative $\varphi'(\sigma)$ and realize the relationship

$$\int_0^\infty \varphi(\sigma)d\sigma = \infty \qquad \int_0^\infty \varphi(\sigma)d\sigma = \infty \qquad [5.11]$$

We note that these conditions are not required in Section 2. It is possible to show, however, that the conditions of [5.11] exist in the given case. In this very case, from the convergence of even one of the integrals in [5.11], there would be an absence of overall stability (5). We underscore the difference from that case where the equation of the open loop system has roots in the left semiplane (see Theorem 2).

Statements proved in Section 5 are combined by us into a theorem.

THEOREM 5 *Assume that in system* [1.2] *matrix* A *has one zero characteristic value and the remainder have negative real parts; function* $\varphi(\sigma)$ *has a continuous derivative satisfying* [1.4]; *integrals* [5.11] *diverge.*

Then:

1 *The derivative of any Lyapunov function of the form* [2.1], *by virtue of system* [1.2], *cannot be negatively definite, but can only be nonpositive. In addition there would be an indispensable fulfillment of relationships in* [5.4 *and* 5.5] (*vectors* y_0, z_0 *satisfy the relationships in* [4.1], *matrix* C *enters the expression for derivative* [5.1]).

2 *With the presence of the Lyapunov function of the indicated form, with minimum degeneration, i.e., when matrix* C *has only one zero characteristic value, or in other words satisfies conditions in* [5.10], *the assigned system* [1.2] *is stable on the whole.*

3 *For the existence of the indicated form of the Lyapunov function, with minimum degeneration, it is sufficient that the system of resolving equations in* [2.19] *has a real solution* u *for the assigned vector* b *and all vectors* b, *sufficiently close to the assigned; it is also necessary and sufficient to have a real solution* u *in system* [2.18], (IV), *if even for one vector* c, *where matrix* G *satisfies conditions in* [5.5 *and* 5.10].

Theorem 6 follows from Points 2 and 3.

THEOREM 6 *Suppose that the conditions in Theorem 5 were fulfilled.*

1 *If Equation* [2.17] (*or system* [2.19]) *has a real solution* u *for the assigned vector* b *and all vectors of* b, *sufficiently close to the assigned, then the trivial solution of system* [1.2] *is stable on the whole.*

2 *If the equations in* [2.18], *where* C *is any matrix, satisfying the conditions in* [5.5 *and* 5.7], *admit solution* u, *then the trivial solution for system* [1.2] *is stable on the whole.*

For $n = 2, 3, 4, 5, 6$ in the presence of one zero root in an equation $\det(A - \lambda I) = 0$, stability conditions can be ob-

[29] Using formulas [2.1–2.4] and the computations in Section 2, these formulas will be obtained most simply.

tained which are analogous to the conditions presented at the end of Section 3; we omit them.

Section 6 Basic Results. Sequence of Computations

System [1.2] is considered, where $\rho > 0$ and $\varphi(\sigma)$; it is a continuous function, satisfying the conditions in [1.4].

1′ The characteristic equation of the open loop system $\det(A - \lambda I) = 0$ has all roots with negative real parts.

The resolving system of the quadratic equations is formed in the following way.

Let $u = \|\xi_j\|_{j=1}^{n}$. Then uu^* is a matrix of the form $uu^* = \|\xi_i\xi_j\|_{i,j=1}^{n}$. Matrix U is determined from the first equation in [2.19] (the solution is unique) and it is substituted in the second equation of [2.19]. The resultant vector equation also provides n quadratic "resolving" equations of the form [3.13].

Γ^2 is determined from formula [3.2]. Condition $\Gamma^2 > 0$ is a necessary condition of stability. Formulas in [3.1 and 3.3] provide one linear relationship between ξ_j. Using this relationship, the order of the system of resolving equations should be reduced by a unit, eliminating one of the ξ_j.

Eliminating variables ξ_j in succession, the system of quadratic equations should lead to an algebraic equation of the form [3.14].

In using one of the methods for separating real roots, it should be established whether this equation has real roots (or, whether the coefficients of the assigned system of differential equations depend on the parameters, and for which values of these parameters this equation has real roots). If this equation has at least two real roots, or in other words, if the system of resolving equations has real solutions, then the undisturbed movement is stable on the whole (Theorem 2, Point 1).

2′ If an equation of the type [3.14] has no real roots, then this is no evidence of instability. In order to obtain broader stability conditions in the resolving system of quadratic equations, additional parameters should be introduced from formulas in [2.18], where C is an arbitrary positively definite matrix (it is possible to take $C = FF^*$ where F is an arbitrary nonsingular matrix). The resolving system of Equation [2.18], (IV) has the same form as the earlier one where elements of vector c would be additional parameters obeying certain inequalities. If the resolving system of the quadratic equations, when we select additional parameters, has a real solution, then the undisturbed movement is stable on the whole. If with any selection of additional parameters the resolving system has no real solutions, then it is either not stable on the whole or it is; however, it cannot be discovered by the Lyapunov function of the form [2.1].

3′ The characteristic equation of the open loop system has the zero root with a multiplicity of $k > 1$. The undisturbed movement is unstable, despite the nature of the remaining roots, even for a linear system (Section 4, 2′).

4′ The characteristic equation of the open loop system has one zero root.

The resolving system of the quadratic equations is formed just as in 1′. Matrix U is determined only if the vector $u = \|\xi_j\|_{j=1}^{n}$ satisfies certain relationships [namely $(y_0, u) = 0$] and differently. This difference has no influence on the final form of the resolving equations. Using the obtained relationship[30] $(y_0, u) = 0$, the order of the system can be reduced by a unit, eliminating one of the ξ_j. Vectors x_0, y_0, z_0 are determined (uniquely) from Equations [4.15, 4.16 and 4.17]; Γ^2 is determined from formula [4.11]. The necessary conditions of stability $\Gamma^2 \geq 0$, $(y_0, z_0) < 0$ and the necessary condition for the existence of a real solution for resolving system is $\Gamma^2 > 0$. In using Equation [4.12] the order of resolving equation systems should be reduced by another unit.

Further computations and conclusions about stability are the same as in Point 1 (Theorem 5).

5′ If the resolving system has no real solutions then additional parameters should be introduced into the resolving system, as in Point 2′. Conclusions of stability are the same as in Point 2′.

[30] This relationship determines vector y_0 only to within a multiple.

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Reviewer's Comment

The basic objective of the paper is to study the stability of the nonlinear control systems given in Equations [1.1 and 1.10] of the paper. We feel that for a better understanding of the paper the block diagram representation of the systems under consideration is desirable.

The author uses the Lyapunov second, direct method, as the basis for the discussion. The results obtained refer to particular systems which in many cases deviate significantly from the practical situations. It is of particular interest, however, that the treatment is rigorous enough to give a basis for the further extensions and applications.

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Perturbations in the Motion of Artificial Satellites Due to the Oblateness of the Earth

V. F. PROSKURIN and
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Literal expressions have been derived for first-order perturbations in the orbit elements of artificial Earth satellites correct to the first power of the Earth's oblateness and up to and including the fifth power of the orbital eccentricity. The coefficients of these expressions depend on the inclination of the orbit by way of trigonometric polynomials. More accurate expressions are given for secular first-order perturbations in the longitude of the node, in the argument of the perigee and in the mean anomaly. The secular motion of the node has been determined taking into account second-order perturbations due to the Earth's oblateness. A numerical example illustrates the comparative magnitude of the perturbations.

THE FACT that the Earth is not a true sphere is one of the most important causes of the deviation of the orbits of artificial satellites from undisturbed Keplerian ellipses, the greatest perturbations in the motion of such satellites being due to the oblateness of the Earth.

The problem of determining the motion of a satellite in the gravitational field of an oblate planet arose earlier in connection with the development of the theory of motion of satellites of large planets. However, the orbits of artificial satellites have a number of characteristics which forbid the use of existing methods in constructing a theory of their motion. Among these are primarily the large orbital inclinations of artificial satellites and their proximity to the surface of the Earth.

It therefore becomes necessary to construct a new analytical theory, suitable for artificial satellites with any orbital inclination to the equatorial plane and sufficiently accurate even for artificial satellites moving in direct proximity to the surface of the Earth.

In this article it is assumed that the planet is an equipotential ellipsoid of revolution and that its oblateness is sufficiently small for its third power to be neglected. The part of the perturbation function in question is expanded in powers of the eccentricity; the coefficients of the series are trigonometric functions of the inclination. Integration of the Lagrange equations gives analytical expressions for the first-order perturbations with respect to flattening of all the orbital elements correct to the fifth power of the eccentricity inclusive. From the second-order perturbations with respect to flattening we get purely secular terms in the motion of the node of the satellite orbit.

1 Formulation of the Problem. Expansion of the Disturbing Function

Let a satellite of zero mass (S in the figure) move in the gravitational field of the Earth, the outer surface of the latter constituting an equipotential ellipsoid of revolution. The oblateness of the Earth and its angular velocity may be considered small quantities. The resistance of the atmosphere is not taken into account.

As is known, correct to the second power of the flattening the potential of such an ellipsoid of revolution for an external point has the form

$$V = \frac{fm}{r}\left[1 - \frac{J}{3}\left(\frac{a'}{r}\right)^2 (3\sin^2\delta - 1) + \frac{D}{35}\left(\frac{a'}{r}\right)^4 \times (35\sin^4\delta - 30\sin^2\delta + 3)\right] = \frac{fm}{r} + R \quad [1]$$

where

f = gravity constant
m = Earth's mass
r = radius vector of point S
a' = Earth's equatorial radius
δ = declination of satellite

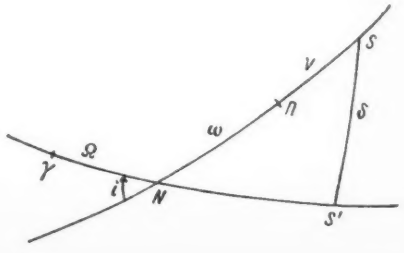
The coefficients J, D are defined by the formulas

$$J = \epsilon - \frac{m}{2} + \epsilon\left(\frac{\epsilon}{2} - \frac{m}{7}\right)$$

$$D = (\epsilon/4)(5J + 9\epsilon)$$

$$m = (\omega^2 a'^3/fm)(1 - \epsilon) \quad [2]$$

where ϵ is the flattening of the terrestrial ellipsoid and ω its angular rotational velocity.



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If in the disturbing function R, defined by formula [1], we confine ourselves to the first power of the flattening, then

$$R = -\frac{1}{3} J \frac{fma'^2}{r^3} (3 \sin^2 \delta - 1) \qquad [3]$$

In order to get an expansion of R in powers of the orbital eccentricity of the satellite, it is convenient first to make certain transformations of expression [3]. From the spherical triangle SNS' we have

$$\sin \delta = \sin i \sin (v + \omega) \qquad [4]$$

where

i = orbital inclination
v = true anomaly
ω = angular distance of perigee from node (argument of perigee)

Substituting Equation [4] in Equation [3] after certain transformations we get

$$R = \frac{1}{6} Jfm \frac{a'^2}{a^3} \left[(2 - 3\lambda^2) \left(\frac{a}{r}\right)^3 + 3\lambda^2 \cos 2\omega \left(\frac{a}{r}\right)^3 \cos 2v - 3\lambda^2 \sin 2\omega \left(\frac{a}{r}\right)^3 \sin 2v \right] \qquad [5]$$

where a is the semimajor axis of the orbit and for conciseness $\lambda = \sin i$. For the combinations $(a/r)^3$, $(a/r)^3 \cos 2v$, $(a/r)^3 \sin 2v$ there are expansions with respect to multiples of the mean anomaly M (Subbotin, 1937). We shall write these expansions up to the sixth power of the eccentricity e inclusive

$$\left(\frac{a}{r}\right)^3 = 1 + \frac{3}{2} e^2 + \frac{15}{8} e^4 + \frac{35}{16} e^6 + \left(3e + \frac{27}{8} e^3 + \frac{261}{64} e^5\right) \cos M +$$
$$+ \left(\frac{9}{2} e^2 + \frac{7}{2} e^4 + \frac{141}{32} e^6\right) \cos 2M + \left(\frac{53}{8} e^3 + \frac{392}{128} e^5\right) \cos 3M +$$
$$+ \left(\frac{77}{8} e^4 + \frac{129}{80} e^6\right) \cos 4M + \frac{1773}{128} e^5 \cos 5M + \frac{3167}{160} e^6 \cos 6M,$$
$$\left(\frac{a}{r}\right)^3 \cos 2v = \left(-\frac{1}{2} e + \frac{1}{12} e^3 + \frac{1}{768} e^5\right) \cos M + \left(1 - \frac{5}{2} e^2 + \right.$$
$$\left. + \frac{41}{48} e^4 - \frac{133}{1440} e^6\right) \cos 2M + \left(\frac{7}{2} e - \frac{123}{16} e^3 + \frac{4971}{1280} e^5\right) \cos 3M +$$
$$+ \left(\frac{17}{2} e^2 - \frac{115}{6} e^4 + \frac{9079}{720} e^6\right) \cos 4M + \left(\frac{845}{48} e^3 + \frac{32\,525}{768} e^5\right) \cos 5M +$$
$$+ \left(\frac{533}{16} e^4 - \frac{13\,827}{160} e^6\right) \cos 6M + \frac{228\,347}{3840} e^5 \cos 7M + \frac{73\,369}{720} e^6 \cos 8M,$$
$$\left(\frac{a}{r}\right)^3 \sin 2v = \left(-\frac{1}{2} e + \frac{1}{24} e^3 - \frac{7}{256} e^5\right) \sin M + \left(1 - \frac{5}{2} e^2 + \right.$$
$$\left. + \frac{37}{48} e^4 - \frac{217}{1440} e^6\right) \sin 2M + \left(\frac{7}{2} e - \frac{123}{16} e^3 + \frac{4809}{1280} e^5\right) \sin 3M +$$
$$+ \left(\frac{17}{2} e^2 - \frac{115}{6} e^4 + \frac{8951}{720} e^6\right) \sin 4M + \left(\frac{845}{48} e^3 - \frac{32\,525}{768} e^5\right) \sin 5M +$$
$$+ \left(\frac{533}{16} e^4 - \frac{13\,827}{160} e^6\right) \sin 6M + \frac{228\,347}{3840} e^5 \sin 7M + \frac{73\,369}{720} e^6 \sin 8M. \qquad (6)$$

Substituting Equation [6] in Equation [3], we get the final expression for R, correct to the sixth power of the eccentricity

$$R = \frac{1}{3} Jfm \frac{a'^2}{a^3} \left(1 - \frac{3}{2} \lambda^2\right) \left[1 + \frac{3}{2} e^2 + \frac{15}{8} e^4 + \frac{35}{16} e^6 + 3 \left(e + \frac{9}{8} e^3 + \frac{87}{64} e^5\right) \cos M + \right.$$
$$+ \frac{9}{2} \left(e^2 + \frac{7}{9} e^4 + \frac{47}{48} e^6\right) \cos 2M + \frac{53}{8} \left(e^3 + \frac{393}{848} e^5\right) \cos 3M + \frac{77}{8} \left(e^4 + \frac{129}{770} e^6\right) \cos 4M +$$
$$\left. + \frac{1773}{128} e^5 \cos 5M + \frac{3167}{160} e^6 \cos 6M\right] + \frac{1}{2} Jfm \frac{a'^2}{a^3} \lambda^2 \left[\frac{1}{48} \left(e^3 + \frac{11}{16} e^5\right) \cos (M - 2\omega) + \right.$$

$$+\frac{1}{24}\left(e^4+\frac{7}{10}e^6\right)\cos(2M-2\omega)+\frac{81}{1280}e^5\cos(3M-2\omega)+\frac{4}{45}e^6\cos(4M-2\omega)-$$

$$-\frac{1}{2}\left(e-\frac{1}{8}e^3+\frac{5}{192}e^5\right)\cos(M+2\omega)+\left(1-\frac{5}{2}e^2+\frac{13}{16}e^4-\frac{35}{288}e^6\right)\cos(2M+2\omega)+$$

$$+\frac{7}{2}\left(e-\frac{123}{56}e^3+\frac{489}{448}e^5\right)\cos(3M+2\omega)+\frac{17}{2}\left(e^2-\frac{115}{51}e^4+\frac{601}{408}e^6\right)\cos(4M+2\omega)+$$

$$+\frac{845}{48}\left(e^3-\frac{6505}{2704}e^5\right)\cos(5M+2\omega)+\frac{533}{16}\left(e^4-\frac{13\,827}{5330}e^6\right)\cos(6M+2\omega)+$$

$$+\frac{228\,347}{3840}e^5\cos(7M+2\omega)+\frac{73\,369}{720}e^6\cos(8M+2\omega)\Big]. \qquad (7)$$

2 First-Order Perturbations

The Lagrange equations for determining the osculating elements have the form (Subbotin, 1937)[1]

$$\begin{aligned}
\frac{da}{dt}&=\frac{2}{na}\frac{\partial R}{\partial\varepsilon},\\
\frac{de}{dt}&=-\frac{\sqrt{1-e^2}}{na^2e}\cdot\frac{\partial R}{\partial\pi}-\frac{e\sqrt{1-e^2}}{1+\sqrt{1-e^2}}\cdot\frac{1}{na^2}\cdot\frac{\partial R}{\partial\varepsilon},\\
\frac{di}{dt}&=-\frac{\operatorname{cosec} i}{na^2\sqrt{1-e^2}}\cdot\frac{\partial R}{\partial\Omega}-\frac{\operatorname{tg}\frac{i}{2}}{na^2\sqrt{1-e^2}}\left(\frac{\partial R}{\partial\pi}+\frac{\partial R}{\partial\varepsilon}\right),\\
\frac{d\Omega}{dt}&=\frac{\operatorname{cosec} i}{na^2\sqrt{1-e^2}}\cdot\frac{\partial R}{\partial i}, \qquad (8)\\
\frac{d\pi}{dt}&=\frac{\operatorname{tg}\frac{i}{2}}{na^2\sqrt{1-e^2}}\cdot\frac{\partial R}{\partial i}+\frac{\sqrt{1-e^2}}{na^2e}\cdot\frac{\partial R}{\partial e},\\
\frac{\partial\varepsilon}{\partial t}&=-\frac{2}{na}\cdot\frac{\partial R}{\partial a}+\frac{\operatorname{tg}\frac{i}{2}}{na^2\sqrt{1-e^2}}\cdot\frac{\partial R}{\partial i}+\frac{e\sqrt{1-e^2}}{1+\sqrt{1-e^2}}\cdot\frac{1}{na^2}\cdot\frac{\partial R}{\partial e},
\end{aligned}$$

where a, e, i, Ω, $\pi=\omega+\Omega$, ϵ are the six orbital elements of the satellite, n is the mean motion and R is the disturbing function.

Note that in the equation for π the derivative $\partial R/\partial e$ is still multiplied by e to a negative power. This leads to a reduction in the accuracy of the expression for first-order perturbations on the element π, so that the expression $e\delta_1\pi$, required for determining the disturbed radius vector and longitude of the satellite, is obtained correct only to the fifth power of eccentricity. The perturbations of the other elements also should be obtained with this accuracy, although they could be obtained up to the sixth power of the eccentricity inclusive.

Expanding coefficients of the right members of Equations [8] in powers of the eccentricity and substituting R from Equation [7], after integrating [8] we get the following expressions for the first-order perturbations of the elliptical orbital elements of the satellite

$$\begin{aligned}
\frac{\delta_1 a}{a}&=2J\left(\frac{a'}{a}\right)^2\left(1-\frac{3}{2}\lambda^2\right)\Big[\left(e+\frac{9}{8}e^3+\frac{87}{64}e^5\right)\cos M+\frac{3}{2}\left(e^2+\frac{7}{9}e^4\right)\cos 2M+\\
&\quad+\frac{53}{24}\left(e^3+\frac{393}{848}e^5\right)\cos 3M+\frac{77}{24}e^4\cos 4M+\frac{591}{128}e^5\cos 5M\Big]+\\
&+J\left(\frac{a'}{a}\right)^2\lambda^2\Big[-\frac{1}{2}\left(e-\frac{1}{8}e^3+\frac{5}{192}e^5\right)\cos(M+2\omega)+\frac{1}{48}\left(e^3+\frac{11}{16}e^5\right)\cos(M-2\omega)+
\end{aligned}$$

[1] Note: In Equation [8] and other equations which follow, tg stands for tan.

$$+\left(1-\frac{5}{2}e^2+\frac{13}{16}e^4\right)\cos(2M+2\omega)+\frac{1}{24}e^4\cos(2M-2\omega)+$$

$$+\frac{7}{2}\left(e-\frac{123}{56}e^3+\frac{489}{448}e^5\right)\cos(3M+2\omega)+\frac{81}{1280}e^5\cos(3M-2\omega)+$$

$$+\frac{17}{2}\left(e^2-\frac{115}{51}e^4\right)\cos(4M+2\omega)+\frac{845}{48}\left(e^3-\frac{6505}{2704}e^5\right)\cos(5M+2\omega)+$$

$$+\frac{533}{16}e^4\cos(6M+2\omega)+\frac{228\,347}{3840}e^5\cos(7M+2\omega)\Big]. \tag{9a}$$

$$\delta_1 e=J\left(\frac{a'}{a}\right)^2\left(1-\frac{3}{2}\lambda^2\right)\Big[\left(1+\frac{1}{8}e^2+\frac{15}{64}e^4\right)\cos M+\frac{3}{2}\left(e-\frac{2}{9}e^3+\frac{29}{144}e^5\right)\cos 2M+$$

$$+\frac{53}{24}\left(e^2-\frac{455}{848}e^4\right)\cos 3M+\frac{77}{24}\left(e^3-\frac{641}{770}e^5\right)\cos 4M+\frac{591}{128}e^4\cos 5M+\frac{3167}{480}e^5\cos 6M\Big]+$$

$$+\frac{1}{2}J\left(\frac{a'}{a}\right)^2\lambda^2\Big[\frac{1}{2}\left(1-\frac{1}{8}e^2-\frac{43}{192}e^4\right)\cos(M+2\omega)+\frac{1}{16}\left(e^2+\frac{1}{48}e^4\right)\cos(M-2\omega)+$$

$$+\frac{1}{12}\left(e^3-\frac{1}{20}e^5\right)\cos(2M-2\omega)-\frac{1}{2}\left(e-\frac{11}{4}e^3+\frac{21}{16}e^5\right)\cos(2M+2\omega)+$$

$$+\frac{7}{6}\left(1-\frac{235}{56}e^2+\frac{2569}{448}e^4\right)\cos(3M+2\omega)+\frac{27}{256}e^4\cos(3M-2\omega)+$$

$$+\frac{17}{4}\left(e-\frac{383}{102}e^3+\frac{254}{51}e^5\right)\cos(4M+2\omega)+\frac{2}{15}e^5\cos(4M-2\omega)+$$

$$\frac{169}{16}\left(e^2-\frac{30\,331}{8112}e^4\right)\cos(5M+2\omega)+\frac{533}{24}\Big(\quad-\frac{40\,979}{10\,660}e^5\Big)\cos(6M+2\omega)+$$

$$+\frac{32\,621}{768}e^4\cos(7M+2\omega)+\frac{73\,369}{960}e^5\cos(8M+2\omega)\Big]. \tag{9b}$$

$$\delta_1 i=\frac{1}{2}J\left(\frac{a'}{a}\right)^2\lambda\sqrt{1-\lambda^2}\Big[-\left(e+\frac{3}{8}e^3+\frac{65}{192}e^5\right)\cos(M+2\omega)-\frac{1}{24}\left(e^3+\frac{19}{16}e^5\right)\cos(M-2\omega)+$$

$$+\left(1-2e^2-\frac{1}{16}e^4\right)\cos(2M+2\omega)-\frac{1}{24}e^4\cos(2M-2\omega)+$$

$$+\frac{7}{3}\left(e-\frac{95}{56}e^3+\frac{165}{448}e^5\right)\cos(3M+2\omega)-\frac{27}{640}e^5\cos(3M-2\omega)+$$

$$+\frac{17}{4}\left(e^2-\frac{179}{102}e^4\right)\cos(4M+2\omega)+\frac{169}{24}\left(e^3-\frac{5153}{2704}e^5\right)\cos(5M+2\omega)+$$

$$+\frac{533}{48}e^4\cos(6M+2\omega)+\frac{32\,621}{1920}e^5\cos(7M+2\omega\ \Big], \tag{9c}$$

$$\delta_1\Omega=-J\left(\frac{a'}{a}\right)^2\sqrt{1-\lambda^2}\Big[(1+2e^2+3e^4)\,nt+3\left(e+\frac{13}{8}e^3+\frac{147}{64}e^5\right)\sin M+$$

$$+\frac{9}{4}\left(e^2+\frac{23}{18}e^4\right)\sin 2M+\frac{53}{24}\left(e^3+\frac{817}{848}e^5\right)\sin 3M+\frac{77}{32}e^4\sin 4M+\frac{1773}{640}e^5\sin 5M+$$

$$+\frac{1}{2}\left(e+\frac{3}{8}e^3+\frac{65}{192}e^5\right)\sin(M+2\omega)-\frac{1}{48}\left(e^3+\frac{19}{16}e^5\right)\sin(M-2\omega)-$$

$$-\frac{1}{2}\left(1-2e^2-\frac{1}{16}e^4\right)\sin(2M+2\omega)-\frac{1}{48}e^4\sin(2M-2\omega)-$$

$$-\frac{7}{6}\left(e-\frac{95}{56}e^3+\frac{165}{448}e^5\right)\sin(3M+2\omega)-\frac{27}{1280}e^5\sin(3M-2\omega)-$$

$$-\frac{17}{8}\left(e^2-\frac{179}{102}e^4\right)\sin(4M+2\omega)-\frac{169}{48}\left(e^3-\frac{5153}{2704}e^5\right)\sin(5M+2\omega)-$$

$$-\frac{533}{96}e^4\sin(6M+2\omega)-\frac{32\,621}{3840}e^5\sin(7M+2\omega)\Big]. \tag{9d}$$

$$e\delta_1\pi=-J\left(\frac{a'}{a}\right)^2(\sqrt{1-\lambda^2}-1+\lambda^2)\Big[(e+2e^3+3e^5)\,nt+3\left(e^2+\frac{13}{8}e^4\right)\sin M+$$

$$+\frac{9}{4}\left(e^3+\frac{23}{18}e^5\right)\sin 2M+\frac{53}{24}e^4\sin 3M+\frac{77}{32}e^5\sin 4M+\frac{1}{2}\left(e^2+\frac{3}{8}e^4\right)\sin(M+2\omega)-$$

$$-\frac{1}{48}e^4\sin(M-2\omega)-\frac{1}{2}\left(e-2e^3-\frac{1}{16}e^5\right)\sin(2M+2\omega)-\frac{1}{48}e^5\sin(2M-2\omega)-$$

$$-\frac{7}{6}\left(e^2-\frac{95}{56}e^4\right)\sin(3M+2\omega)-\frac{17}{8}\left(e^3-\frac{179}{102}e^5\right)\sin(4M+2\omega)-\frac{169}{48}e^4\sin(5M+2\omega)-$$

$$-\frac{533}{96}e^5\sin(6M+2\omega)\Big]+J\left(\frac{a'}{a}\right)^2\left(1-\frac{3}{2}\lambda^2\right)\Big[(e+2e^3+3e^5)\,nt+\left(1+\frac{23}{8}e^2+\frac{319}{64}e^4\right)\sin M+$$

$$+\frac{3}{2}\left(e+\frac{19}{18}e^3+\frac{293}{144}e^5\right)\sin 2M+\frac{53}{24}\left(e^2+\frac{231}{848}e^4\right)\sin 3M+\frac{77}{24}\left(e^3-\frac{383}{1540}e^5\right)\sin 4M+$$

$$+\frac{591}{128}e^4\sin 5M+\frac{3167}{480}e^5\sin 6M\Big]-\frac{1}{4}J\left(\frac{a'}{a}\right)^2\lambda^2\Big[\left(1-\frac{7}{8}e^2+\frac{37}{192}e^4\right)\sin(M+2\omega)-$$

$$-\frac{1}{8}\left(e^2+\frac{31}{48}e^4\right)\sin(M-2\omega)+5\left(e-\frac{23}{20}e^3+\frac{83}{240}e^5\right)\sin(2M+2\omega)-$$

$$-\frac{1}{6}\left(e^3+\frac{11}{20}e^5\right)\sin(2M-2\omega)-\frac{27}{128}e^4\sin(3M-2\omega)-$$

$$-\frac{7}{3}\left(1-\frac{397}{56}e^2+\frac{3865}{448}e^4\right)\sin(3M+2\omega)-\frac{4}{15}e^5\sin(4M-2\omega)-$$

$$-\frac{17}{2}\left(e-\frac{511}{102}e^3+\frac{334}{51}e^5\right)\sin(4M+2\omega)-\frac{169}{8}\left(e^2-\frac{36\,581}{8112}e^4\right)\sin(5M+2\omega)-$$

$$-\frac{533}{12}\left(e^3-\frac{46\,811}{10\,660}e^5\right)\sin(6M+2\omega)-\frac{32\,621}{384}e^4\sin(7M+2\omega)-\frac{73\,369}{480}e^5\sin(8M+2\omega)\Big],\quad(9e)$$

$$\delta_1\varepsilon=2J\left(\frac{a'}{a}\right)^2\left(1-\frac{3}{2}\lambda^2\right)\Big[\left(1+\frac{7}{4}e^2+\frac{39}{16}e^4\right)nt+\frac{13}{4}\left(e+\frac{133}{104}e^3+\frac{109}{64}e^5\right)\sin M+$$

$$+\frac{21}{8}\left(e^2+\frac{215}{522}e^4\right)\sin 2M+\frac{265}{96}\left(e^3+\frac{403}{848}e^5\right)\sin 3M+\frac{77}{24}e^4\sin 4M+\frac{10\,047}{2560}e^5\sin 5M\Big]+$$

$$+3J\left(\frac{a'}{a}\right)^2\lambda^2\Big[-\frac{13}{24}\left(e-\frac{17}{104}e^3+\frac{79}{2496}e^5\right)\sin(M+2\omega)+\frac{5}{192}\left(e^3+\frac{35}{48}e^5\right)\sin(M-2\omega)+$$

$$+\frac{1}{2}\left(1-\frac{35}{12}e^2+\frac{19}{16}e^4\right)\sin(2M+2\omega)+\frac{1}{36}e^4\sin(2M-2\omega)+$$

$$+\frac{91}{72}\left(e-\frac{143}{56}e^3+\frac{8995}{5824}e^5\right)\sin(3M+2\omega)+\frac{153}{5120}e^5\sin(3M-2\omega)+$$

$$+\frac{119}{48}\left(e^2-\frac{533}{204}e^4\right)\sin(4M+2\omega)+\frac{845}{192}\left(e^3-\frac{112\,613}{40\,560}e^5\right)\sin(5M+2\omega)+$$

$$+\frac{533}{72}e^4\sin(6M+2\omega)+\frac{554\,557}{46\,080}e^5\sin(7M+2\omega)\Big]-$$

$$-J\left(\frac{a'}{a}\right)^2\left(\sqrt{1-\lambda^2}-1+\lambda^2\right)\Big[(1+2e^2+3e^4)\,nt+3\left(e+\frac{13}{8}e^3+\frac{147}{64}e^5\right)\sin M+$$

$$+\frac{9}{4}\left(e^2+\frac{23}{18}e^4\right)\sin 2M+\frac{53}{24}\left(e^3+\frac{817}{848}e^5\right)\sin 3M+\frac{77}{32}e^4\sin 4M+\frac{1773}{640}e^5\sin 5M+$$

$$+\frac{1}{2}\left(e+\frac{3}{8}e^3+\frac{65}{192}e^5\right)\sin(M+2\omega)-\frac{1}{48}\left(e^3+\frac{19}{16}e^5\right)\sin(M-2\omega)-$$

$$-\frac{1}{2}\left(1-2e^2-\frac{1}{16}e^4\right)\sin(2M+2\omega)-\frac{1}{48}e^4\sin(2M-2\omega)-$$

$$-\frac{7}{6}\left(e-\frac{95}{56}e^3+\frac{165}{448}e^5\right)\sin(3M+2\omega)-\frac{27}{1280}e^5\sin(3M-2\omega)-$$

$$-\frac{17}{8}\left(e^2-\frac{179}{102}e^4\right)\sin(4M+2\omega)-\frac{169}{48}\left(e^3-\frac{5153}{2704}e^5\right)\sin(5M+2\omega)-\frac{533}{96}e^4\sin(6M+2\omega)-$$

$$-\frac{32\,621}{3840}e^5\sin(7M+2\omega)\Big]\qquad(9f)$$

The perturbations of the elements obtained may be used to compute the disturbed positions of artificial satellites. For this purpose we employ the formulas

$$M=M_0+n_0(t-t_0)+\delta_1\varepsilon-\delta_1\pi-\frac{3}{2}\frac{n_0}{a_0}\int_{t_0}^{t}\delta_1 a\,dt,$$

$$E=M+e\sin E,$$

$$r=a(1-e\cos E),$$

$$\operatorname{tg}\frac{v}{2}=\sqrt{\frac{1+e}{1-e}}\operatorname{tg}\frac{E}{2},$$

$$u=v+\omega,$$

$$x=r(\cos u\cos\Omega-\sin u\sin\Omega\cos i),$$

$$y=r(\cos u\sin\Omega+\sin u\cos\Omega\cos i),$$

$$z=r\sin u\sin i,$$

where the letters with zero subscripts indicate the undisturbed values of the elements. Note that there are secular perturbations only in the elements Ω, π, ϵ. They are absent from the remaining elements.

3 Secular Perturbations of the First Order

Among the first-order perturbations the most important are the secular perturbations, since they determine the evolution of the orbit in the course of time. Therefore it is useful to know the secular perturbations with high accuracy.

The special structure of the disturbing function makes it possible to obtain the secular perturbations in finite form, without resorting to expansion in powers of the eccentricity.

In computing the secular perturbations it is convenient to use instead of the elements π, ϵ the elements ω, M_0, given by the formulas

$$\omega = \pi - \Omega \qquad M_0 = \epsilon - \pi$$

The Lagrange equations for the elements ω, M_0 have the form

$$\frac{d\omega}{dt} = -\frac{\cot i}{na^2\sqrt{1-e^2}}\frac{\partial R}{\partial i} + \frac{\sqrt{1-e^2}}{na^2e} \times \frac{\partial R}{\partial e}$$

$$\frac{dM_0}{dt} = -\frac{1-e^2}{na^2e} \times \frac{\partial R}{\partial e} - \frac{2}{na} \times \frac{\partial R}{\partial a} \qquad \text{[10]}$$

We shall denote the coefficients in the secular perturbations of the elements Ω, ω, M_0 by Ω', ω', M_0' respectively. Then as independent variable we shall introduce the true anomaly v, linked with t by the formula

$$r^2dv = \sqrt{fm}\sqrt{a(1-e^2)}dt$$

Then we shall have

$$\omega' = \frac{1}{2\pi\sqrt{fm}\sqrt{a(1-e^2)}}\int_0^{2\pi} r^2\frac{d\omega}{dt}\,dv$$

$$\Omega' = \frac{1}{2\pi\sqrt{fm}\sqrt{a(1-e^2)}}\int_0^{2\pi} r^2\frac{d\Omega}{dt}\,dv$$

$$M_0' = \frac{1}{2\pi\sqrt{fm}\sqrt{a(1-e^2)}}\int_0^{2\pi} r^2\frac{dM_0}{dt}\,dv \qquad \text{[11]}$$

Formulas [11] give the coefficients Ω', ω', M_0' on the assumption that the unit of time is chosen as the quantity $T/2\pi$, where T is the period of revolution of the satellite about the Earth. If it is desirable to take 24 hr as the unit of time measurement, then it is necessary to multiply the right members of formulas [11] by the mean diurnal motion of the satellite n, the coefficients Ω', ω', M_0' then being obtained in radians or degrees according to whether n is expressed in radians or degrees.

After multiplication of the right members by n formulas [11] give[2]

$$\Omega' = -J\left(\frac{a'}{a}\right)^2\frac{\cos i}{(1-e^2)^2}n$$

$$\omega' = \frac{1}{2}J\left(\frac{a'}{a}\right)^2\frac{5\cos^2 i - 1}{(1-e^2)^2}n$$

$$M_0' = \frac{1}{2}J\left(\frac{a'}{a}\right)^2\frac{3\cos^2 i - 1}{(1-e^2)^{3/2}}n \qquad \text{[12]}$$

It is now easy to formulate expressions for the coefficients in secular perturbations of π and ϵ. We have

$$\pi' = \frac{1}{2}J\left(\frac{a'}{a}\right)^2\frac{5\cos^2 i - 2\cos i - 1}{(1-e^2)^2}n$$

$$\epsilon' = \frac{1}{2}J\left(\frac{a'}{a}\right)^2 \times \frac{(5+3\sqrt{1-e^2})\cos^2 i - 2\cos i - 1 - \sqrt{1-e^2}}{(1-e^2)^2}n \qquad \text{[13]}$$

It is evident from Equation [12] that the secular motion of the node reaches a maximum when $i = 0, 180°$, i.e. when the orbit lies in the equatorial plane. When $i = 90°$, i.e., when the orbital plane passes through the pole, $\Omega' = 0$. The secular motion of the perigee reaches a maximum when $i = 0$, $180°$ and vanishes when $i = 63° 26'$. The secular motion of the element M_0 reaches a maximum when $i = 0, 180°$ and vanishes when $i = 54° 44'$.

4 Example of Computation of First-Order Perturbations

To illustrate the computation of perturbations we shall take an orbit with the following elements

$$a = 7286.88 \text{ km}$$
$$e = 0.099493$$
$$i = 65\overset{\circ}{.}4900$$

Further, we shall take the following values of the constants a' and J

$$a' = 6378.39 \text{ km}$$
$$J = 0.00164147$$

After substituting these values of the orbital elements and the constants a' and J in Equations [9], we get the following final expressions for the first-order perturbations due to the oblateness of the Earth

$$\delta_1 a = -0.446\cos M - 0.066\cos 2M - 0.010\cos 3M - 0.001\cos 4M - \\ -0.376\cos(M+2\omega) + 7.401\cos(2M+2\omega) + \\ +2.584\cos(3M+2\omega) + 0.624\cos(4M+2\omega) + \\ +0.131\cos(5M+2\omega) + 0.024\cos(6M+2\omega),$$

$$\delta_1 e = -0.0003046\cos M - 0.0000454\cos 2M - 0.0000066\cos 3M - \\ -0.0000002\cos 4M + 0.0002601\cos(M+2\omega) + \\ +0.0000003\cos(M-2\omega) - 0.0000251\cos(2M+2\omega) + \\ +0.0005824\cos(3M+2\omega) + 0.0002119\cos(4M+2\omega) + \\ +0.0000524\cos(5M+2\omega) + 0.0000113\cos(6M+2\omega) + \\ +0.0000020\cos(7M+2\omega),$$

[2] Expressions for the coefficients Ω', ω' have been obtained in the paper by Okhotsimskiy, Eneyev and Taratynova, 1957.

$$\delta_1 i = -0^{\circ}.00136 \cos(M + 2\omega) + 0^{\circ}.01333 \cos(2M + 2\omega) +$$
$$+ 0^{\circ}.00310 \cos(3M + 2\omega) + 0^{\circ}.00562 \cos(4M + 2\omega) +$$
$$+ 0^{\circ}.00009 \cos(5M + 2\omega) + 0^{\circ}.00001 \cos(6M + 2\omega),$$

$$\delta_1 \Omega = -2^{\circ}.67416t - 0^{\circ}.00907 \sin M - 0^{\circ}.00067 \sin 2M -$$
$$- 0^{\circ}.00007 \sin 3M - 0^{\circ}.00001 \sin 4M - 0^{\circ}.00149 \sin(M + 2\omega) +$$
$$+ 0.01465 \sin(2M + 2\omega) + 0^{\circ}.00341 \sin(3M + 2\omega) +$$
$$+ 0^{\circ}.00062 \sin(4M + 2\omega) + 0^{\circ}.00010 \sin(5M + 2\omega) +$$
$$+ 0^{\circ}.00002 \sin(6M + 2\omega),$$

$$\delta_1 \varepsilon = -4^{\circ}.77956t - 0^{\circ}.01625 \sin M - 0^{\circ}.00154 \sin 2M -$$
$$- 0^{\circ}.00016 \sin 3M - 0^{\circ}.00002 \sin 4M - 0^{\circ}.00894 \sin(M + 2\omega) +$$
$$+ 0^{\circ}.09471 \sin(2M + 2\omega) + 0^{\circ}.02376 \sin(3M + 2\omega) +$$
$$+ 0^{\circ}.00461 \sin(4M + 2\omega) + 0^{\circ}.00080 \sin(5M + 2\omega) +$$
$$+ 0^{\circ}.00014 \sin(6M + 2\omega),$$

$$e\delta_1 \pi = -0^{\circ}.31079t - 0^{\circ}.01846 \sin M - 0^{\circ}.00267 \sin 2M -$$
$$- 0^{\circ}.00038 \sin 3M - 0^{\circ}.00005 \sin 4M - 0^{\circ}.00001 \sin 5M -$$
$$- 0^{\circ}.01488 \sin(M + 2\omega) - 0^{\circ}.00649 \sin(2M + 2\omega) +$$
$$+ 0^{\circ}.00002 \sin(M - 2\omega) + 0^{\circ}.03259 \sin(3M + 2\omega) +$$
$$+ 0^{\circ}.01203 \sin(4M + 2\omega) + 0^{\circ}.00299 \sin(5M + 2\omega) +$$
$$+ 0^{\circ}.00062 \sin(6M + 2\omega) + 0^{\circ}.00012 \sin(7M + 2\omega) +$$
$$+ 0^{\circ}.00002 \sin(8M + 2\omega).$$

The coefficients of the periodic perturbations of the semimajor axis are given in kilometers and in the secular perturbations of the elements Ω, ϵ, π the time must be taken in mean solar days.

The numerical values of the perturbations presented in the foregoing give a graphic idea of their magnitudes.

Thus, in the semimajor axis the most important perturbations are the periodic ones with the arguments $(2M + 2\omega)$ and $(3M + 2\omega)$; their amplitudes are equal to 7.4 and 2.6 km. The most significant perturbation in the eccentricity has the argument $(3M + 2\omega)$ and may give a 4.2-km deviation in the height of the perigee.

The largest periodic perturbations in the other elements are of the order of several minutes of arc. Thus, in all the elements it is necessary to take into account fairly large perturbations.

5 Secular Perturbations of the Second Order in the Motion of the Node

The secular motion of the node is the most important characteristic of the motion of artificial satellites, since, on the one hand, it can be determined from observations with great accuracy and, on the other, it can be used to get a refined value of the oblateness of the Earth. Accordingly, we need to know the theoretical expression for the secular motion of the node with the greatest possible accuracy. We shall now present computations of secular perturbations of the second order with respect to flattening in the motion of the node.

The longitude of the ascending node is determined from the equation

$$\frac{d\Omega}{dt} = \frac{\operatorname{cosec} i}{na^2\sqrt{1 - e^2}} \times \frac{\partial R}{\partial i} \qquad [14]$$

where R is determined from formula [7].

We shall denote the right member of the equation for Ω by Q and put it in the form

$$Q(a, e, i, \omega, M) = Na^{-7/2}\sqrt{1 - \lambda^2}\,[-1 - 3e\cos M + \cos(2M + 2\omega) - (1/2)e\cos(M + 2\omega) + (7/2)e\cos(3M + 2\omega)] \qquad [15]$$

where $N = J\sqrt{fm}\,a'^2$ is a coefficient depending on the constants linked with the Earth.

In Equation [15] we have confined ourselves to the first power of the eccentricity, in order to obtain secular perturbations correct to the zero power of the eccentricity.

Expanding the right member of the equation for Ω in powers of the perturbations of the elements we have

$$\frac{d\Omega}{dt} = Q(a, e, i, \omega, M) =$$
$$Q + \frac{\partial Q}{\partial a}\delta_1 a + \frac{\partial Q}{\partial e}\delta_1 e + \frac{\partial Q}{\partial i}\delta_1 i + \frac{\partial Q}{\partial \omega}\delta_1 \omega + \frac{\partial Q}{\partial M}\delta_1 M \qquad [16]$$

In the right member of Equation [16] we must substitute the undisturbed values of the elements in Q and also in its derivatives. The right member (except for the term Q which after integration gives δ_1, Ω) after the corresponding transformations and integration will give perturbations of the second order with respect to flattening in Ω.

Since Equation [9] does not contain expressions for the perturbations $\delta\omega$, we make use of the known relationships

$$\delta M = \delta\epsilon - \delta\pi - \frac{3}{2}\frac{n}{a}\int \delta a\,dt$$

$$\frac{\partial Q}{\partial \Omega} = -\frac{\partial Q}{\partial \omega} \qquad \frac{\partial Q}{\partial \epsilon} = \frac{\partial Q}{\partial M}$$

$$\omega = \pi - \Omega \qquad \frac{\partial Q}{\partial \pi} = -\left(\frac{\partial Q}{\partial M} + \frac{\partial Q}{\partial \Omega}\right)$$

..... [17]

Putting $\Omega = \Omega_0 + \delta_1\Omega + \delta_2\Omega + \ldots$, where Ω_0 is a constant and $\delta_i\Omega$ are perturbations of the ith order, we obtain from Equation [16], taking into account Equation [17]

$$\delta_2\Omega = \int \left[\frac{\partial Q}{\partial a}\delta_1 a + \frac{\partial Q}{\partial e}\delta_1 e + \frac{\partial Q}{\partial i}\delta_1 i + \frac{\partial Q}{\partial \Omega}\delta_1\Omega + \frac{\partial Q}{\partial \epsilon}\delta_1\epsilon + \frac{\partial Q}{\partial \pi}\delta_1\pi - \frac{3}{2}\frac{n}{a}\frac{\partial Q}{\partial M}\int \delta_1 a\,dt\right] dt$$

Evaluation of the individual integrals gives the following secular terms

$$\int \frac{\partial Q}{\partial a}\delta_1 a\,dt = -\frac{7}{4}J^2\left(\frac{a'}{a}\right)^4 \lambda^2\sqrt{1-\lambda^2}\,nt,$$

$$\int \frac{\partial Q}{\partial i}\delta_1 i\,dt = -\frac{1}{4}J^2\left(\frac{a'}{a}\right)^4 \lambda^2\sqrt{1-\lambda^2}\,nt,$$

$$\int \frac{\partial Q}{\partial \Omega}\delta_1\Omega\,dt = \frac{1}{2}J^2\left(\frac{a'}{a}\right)^4 (1-\lambda^2)\,nt,$$

$$\int \frac{\partial Q}{\partial e}\delta_1 e\,dt = \frac{1}{2}J^2\left(\frac{a'}{a}\right)^4 \sqrt{1-\lambda^2}\left(1-\frac{3}{2}\lambda^2\right)\left(-3-\frac{1}{2}\cos 2\omega\right)nt + \frac{1}{8}J^2\left(\frac{a'}{a}\right)^4 \lambda^2\sqrt{1-\lambda^2}\left(-3\cos 2\omega + \frac{23}{3}\right)nt,$$

$$\int \frac{\partial Q}{\partial \pi}\delta_1\pi\,dt = \frac{1}{2}J^2\left(\frac{a'}{a}\right)^4 \sqrt{1-\lambda^2}\left(1-\frac{3}{2}\lambda^2\right)\left(-3+\frac{1}{2}\cos 2\omega\right)nt - \frac{1}{8}J^2\left(\frac{a'}{a}\right)^4 \lambda^2\sqrt{1-\lambda^2}\left(-3\cos 2\omega - \frac{23}{3}\right)nt,$$

$$\int \frac{\partial Q}{\partial \epsilon}\delta_1\epsilon\,dt = -\frac{3}{2}J^2\left(\frac{a'}{a}\right)^4 \lambda^2\sqrt{1-\lambda^2}\,nt - \frac{1}{2}J^2\left(\frac{a'}{a}\right)^4 \sqrt{1-\lambda^2}\left(\sqrt{1-\lambda^2}-1+\lambda^2\right)nt,$$

$$\int\left[\frac{3}{2}\cdot\frac{n}{a}\cdot\frac{\partial Q}{\partial M}\int \delta_1 a\,dt\right]dt = \frac{3}{4}J^2\left(\frac{a'}{a}\right)^4 \lambda^2\sqrt{1-\lambda^2}\,nt.$$

Hence we obtain the following expression for the coefficient of secular perturbation in the motion of the node of the order of the square of the coefficient J

$$\Omega_{J^2}' = J^2\left(\frac{a'}{a}\right)^4 \sqrt{1-\lambda^2}\left(\frac{19}{6}\lambda^2 - \frac{5}{2}\right)n \qquad [18]$$

It is also easy to obtain the coefficient of secular perturbation in the longitude of the node in the first power with respect to the quantity D. For this we must use the fourth equation of system [8] and take as the disturbing function the expression

$$R = \frac{1}{35}D\frac{fma'^4}{r^5}(35\sin^4\delta - 30\sin^2\delta + 3)$$

Computations give the following expression for the secular motion of the node due to this term of the disturbing function

$$\Omega_D' = D\left(\frac{a'}{a}\right)^4 \sqrt{1-\lambda^2}\left(\frac{3}{2}\lambda^2 - \frac{6}{7}\right)n \qquad [19]$$

Summing Ω_J', Ω_{J^2}' and Ω_D',[3] we get the total secular motion of the ascending node taking into account terms of the second order with respect to the flattening of the Earth[4]

$$\Omega' = -J\left(\frac{a'}{a}\right)^2 \frac{\cos i}{(1-e^2)^2}n + J^2\left(\frac{a'}{a}\right)^4 \cos i \times \left(\frac{19}{6}\sin^2 i - \frac{5}{2}\right)n + D\left(\frac{a'}{a}\right)^4 \cos i\left(\frac{3}{2}\sin^2 i - \frac{6}{7}\right)n \qquad [20]$$

where terms with the coefficients J^2 and D are accurate only to the first power of the orbital eccentricity.

Expression [20] makes it possible to determine the flattening of the terrestrial spheroid, if the observed motion of the node is known.

6 Example of the Determination of the Earth's Oblateness From the Observed Motion of the Node

As an example of the application of the theoretical expression obtained in the foregoing for the motion of the node, we shall compute the flattening of the Earth, using the elements of the second Soviet artificial satellite, given by King-Hele and Merson, 1958, and the observed diurnal motion of the node for the epoch 1958 January 4.0,taken from the paper by King-Hele, 1958. Since we do not have elements for the date January 4.0, we shall make a linear interpolation between two systems of elements lying close to this date. We get the following initial data

Epoch 1958 January 4.0

P	=	100.503 m (Draconitic period)
a	=	7161.19 km
e	=	0.0802
i	=	65.29°
Ω'	=	2.814°

We shall make the semimajor axis of the terrestrial ellipsoid a' equal to 6378.10 km. In computing terms of the second order, where approximate values of the coefficients J and D are sufficient, we shall take

$$J = 0.001637 \qquad D = 0.0000106$$

The mean motion n, entering into Equation [20], is determined from the formula

$$n = 360°1440^m/P \qquad [21]$$

Strictly speaking, we ought to take a sidereal, not a Draconitic period, but the resulting error is small and may be neglected.

After substituting all these data in Equation [20] we get the following value for the coefficient J

$$J = 0.001628$$

[3] Here, for the sake of uniformity, Ω_J' denotes the coefficient of secular perturbation of the longitude of the node of the first order with respect to J.

[4] An analogous formula has been obtained by King-Hele, 1958, but for non-osculating elements.

The flattening of the Earth ϵ is found from the formula

$$\epsilon = J + \frac{m}{2} + \frac{\epsilon^2}{2} - \frac{\epsilon m}{7}$$

putting $m = 3449.79 \times 10^{-6}$. The reciprocal of the Earth's flattening is then

$$1/\epsilon = 297.9$$

A more careful evaluation of the mean motion ought to increase this figure somewhat.

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Reviewer's Comment

The treatment of satellite perturbations as given by Proskurin and Batrakov is based on classical equations of Lagrange. The procedure followed is quite standard. It can be pointed out that the results obtained by celestial mechanicians in this country are more general and can be adapted for computer use more easily than the results given in this paper. For this reason the results given are not likely to be used in this country.

On the basis of this paper it appears that the Russians just started work on general theories of artificial satellites.

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Stability of a Console Cylindrical Shell Under Bending by a Transverse Force With Twisting and Internal Pressure

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DURING investigations into the strength of certain structures the need arises to evaluate the stability of cylindrical shells under various kinds of loads, particularly normal pressure, twisting moments and transverse bending forces.

The general approximate method of such evaluation is indicated in (1).[1] In a subsequent work (2) this method was improved and considerably simplified for the case of simultaneous action of twisting moments and pressure on a cylindrical shell.

This work aims to simplify the evaluation of a cylindrical shell's stability under the simultaneous action of twisting moments, internal pressure and transverse boundary forces. Conditions are shown under which the stability of a shell may be evaluated in the case under investigation by very simple formulas. As a supplement, the case is considered in which a transverse bending force, uniformly distributed along the shell, is added to the aforementioned loads. Experimental results are presented which agree with theoretical results.

The terminology and designations are those accepted in (1 and 2).

Translated from *Design of Three-Dimensional Structures*, no. 5, 1959, pp. 431–449. Translated by Primary Sources, New York.

[1] Numbers in parentheses indicate References at end of paper.

Section 1 Formulation of the Problem and Its Approximate Solution

Let a console cylindrical shell of "average" length [see (2), p. 138], with a rigid ring at its free end, be acted upon by transverse force $Q = \lambda Q_*$ and twisting moment $M = \lambda M_*$, applied to the ring and also by internal pressure $q = \lambda q_*$, uniformly distributed throughout the entire shell. Quantities Q_*, M_* and q_* are taken as given (fixed), so that what is involved is simply the load.

The object is to determine the critical value of λ_{cr} [see (2), p. 137] of the load parameter $\lambda(\lambda \geq 0)$.

We note that it is easy to go from the case of a simple load to the case in which $Q = \lambda Q_*$, $M = M_*$ and $q = q_*$, i.e., when the twisting moment and pressure are fixed and only transverse force Q changes. Some design equations, derived as a result of this transition, are given in what follows.

Let ξR and φR be axial and circumferential coordinates on the surface center of the shell, reading from the free end of the shell. We will assume that the direction of force $Q(Q \geq 0)$ coincides with the direction from point $\xi = 0$. $\varphi = 0$ to point $\xi = 0, \varphi = \pi$.

The results presented in (1) permit the approximate solution of the problem posed. This solution is as follows.

It is held that stability loss in a shell (bulging) may occur either in the generatrix zone $\varphi = \pi$ (designated zone 1), or in any of the generatrix zones $\varphi = \pm(\pi/2)$ (designated zone 2). If $M_* < 0$ (i.e., when the twisting moment, applied to the ring, is from the loaded end side of the shell, in a clockwise direction), then zone 2 is to be understood as the generatrix zone $\varphi = (\pi/2)$; if, however, $M_* > 0$, then the generatrix zone $\varphi = -(\pi/2)$. When $M_* = 0$, any of the indicated zones may be taken as zone 2.

The loss of stability in zone 1 is considered as if this zone were a part of the shell subjected to compressive force $P = -Ql/\pi R^2 = -\lambda Q_* l/\pi R^2$, twisting moment $M = \lambda M_*$ and internal pressure $q = \lambda q_*$. We will denote the corresponding critical value of λ as $\lambda_{cr}^{(1)}$. It can be found with the aid of the diagrammatic method indicated, for example, in (1), pp. 89–90, 105–106.

The stability loss in zone 2 is viewed as if this zone were part of a shell, loaded by torsional moment $\tilde{M} = \pm(2RQ + |M|) = \pm\lambda(2RQ_* + |M|_*) = \lambda\tilde{M}_*$ [the $-$ sign corresponds to generatrix zone $\varphi = (\pi/2)$, the $+$ sign corresponds to generatrix zone $\varphi = -(\pi/2)$] and by internal pressure $Q = \lambda q_*$. We designate the corresponding critical value of λ by $\lambda_{cr}^{(2)}$. It may be found in the manner indicated in (2) with curve Λ presented therein.

In the considered case, when the load consists of $Q = \lambda Q_*$, $M = \lambda M_*$ and $q = \lambda q_*$, the unknown critical value $\lambda(\lambda_{cr})$ would be

$$\lambda_{cr} = \min\,(\lambda_{cr}^{(1)}, \lambda_{cr}^{(2)})^* \qquad [1]$$

[min $(a_1, a_2, \ldots, a_n)$ denotes the least of magnitudes $a_1, a_2, \ldots, a_n$].

In addition, the case $\lambda_{cr}^{(2)} < \lambda_{cr}^{(1)}$ denotes a stability loss (wave formation) in zone 2 from shear stresses, caused by force Q and moment M, while case $\lambda_{cr}^{(1)} < \lambda_{cr}^{(2)}$ denotes the stability loss in zone 1 from axial compression forces, caused by force Q and of shear stresses, caused by moment M.

Generally speaking, stability loss in shells, with the load under consideration, can also arise in the zone which is intermediate in relation to the indicated, let us say close to generatrix $\varphi = (1/4)\pi$ or $\varphi = (5/4)\pi$. In this case, the critical value $\lambda = \lambda_{cr}^{(3)}$ should be less than $\lambda_{cr}^{(1)}$ and $\lambda_{cr}^{(2)}$; however, confining oneself, for simplicity, to the case where $M_* = 0$, $q_* = 0$, it is possible to see, from the formula for the characteristic value of λ (see (1), formula [5.86]) that

$$\lambda_{cr}^{(3)} \geq \sqrt{2}\,\frac{1}{(1/\lambda_{cr}^{(1)})\,(1/\lambda_{cr}^{(2)})}$$

and, consequently

$$\lambda_{cr}^{(3)} \geq (1/\sqrt{2})\,\min\,(\lambda_{cr}^{(1)}, \lambda_{cr}^{(2)})$$

For that reason, $\lambda_{cr}^{(3)}$ can be less than $\lambda_{cr}^{(1)}$ and $\lambda_{cr}^{(2)}$ only when $\lambda_{cr}^{(1)}$ and $\lambda_{cr}^{(2)}$ differ little from each other; moreover, even in this case $\lambda_{cr}^{(3)}$ cannot be too much smaller than the least of $\lambda_{cr}^{(1)}$ and $\lambda_{cr}^{(2)}$ (in an extreme case, where $\lambda_{cr}^{(1)} = \lambda_{cr}^{(2)} = \lambda_1^*$ in conformance with the last inequality, $\lambda_{cr}^{(3)}$ cannot be less than $(1/\sqrt{2})\lambda^* = 0.7\lambda^*$).

Thus, speaking more strictly, evaluating stability on the basis of a consideration of zones 1 and 2 alone refers to the ordinary case when $\lambda_{cr}^{(1)}$ and $\lambda_{cr}^{(2)}$ are substantially different. When $\lambda_{cr}^{(1)} \approx \lambda_{cr}^{(2)}$ this evaluation can lead to a somewhat overestimated value of λ_{cr}.

Further, the indicated stability evaluation was connected with the hypothesis that in the gap between two neighboring nodal lines $\cos\varphi$ changes little, if the wave formation in zone 1 is kept in mind, or $\sin\varphi$, if the wave formation in zone 2 is kept in mind. In other words, for the sufficient accuracy of Equation [1], the distance between two neighboring nodal lines and their angle of inclination to the shell axis should be small enough.

However, one exception has to be made with relation to the nodal lines in zone 1; namely, if in defining $\lambda_{cr}^{(1)}$ it turns out that it corresponds to the value $n_{cr} = 0$, i.e., it is obtained as if only stress P is active, while M and q are absent,[2] then this result should be considered correct, although in this case one does not have to speak of the closeness of the nodal lines. This statement is based on the following. Evidently, equality $n_{cr} = 0$ denotes that with the considered load replacing the actual, the influence of M together with q is slight (either because of the smallness of M_* or thanks to the considerable compensating action of q_*, as regards M_*) and they can be disregarded. But, it is then natural to consider that even with the actual load, the combined influence of M and q on the stability of zone 1 (where the actual load differs from the interchangeable one only by the fact that instead of a uniform compression by forces P, compression is effected by forces $-P\cos\varphi$) is negligible. Therefore, considering the stability of zone 1, it is possible to consider that the actual load for it consists of only the force $-P\cos\varphi$ (φ fluctuates close to value $\varphi = \pi$); as is well known, in this case, P_{cr} differs little from P_{cr}, where there is a uniform compression by force P (3). That is the justification of the aforementioned statement.

We note that in tests when bulging occurred in zone 1 and also, where $n_{cr} = 0$ for interchangeable loads in this zone, that the bulk of the bulging surface had the same form as when there was an axisymmetrical loss of stability (see, further, Fig. 7).

As a supplement to what has been noted about nodal lines, it should be pointed out that their angles of inclination are known to be less than the angles of inclination for nodal lines in zone 2; there, the shear stresses generate oblique waves (larger than $|\tilde{M}| > |M|$). Therefore, if it is established that the angle of inclination of nodal lines in zone 2 is small enough (so that $\sin\varphi$ doesn't change much along the nodal line), then the angle of inclination for the nodal lines in zone 1 would also be small enough (i.e., $\cos\varphi$ wouldn't change much along these lines).

Turning now to zone 2, it is possible to affirm that the required closeness of nodal lines was secured in this zone since, under the action of twisting moment $\tilde{M}$, shells of medium length lose stability with the formation of a sufficiently large number of circumferential waves. The presence of internal pressure does not change this situation [see (2 and 4)].

It follows from the foregoing that there are two conditions under which the indicated estimations of shell stability can be employed with sufficient accuracy:

1 $\lambda_{cr}^{(2)}$ should correspond to the small wave inclination in zone 2.

2 $\lambda_{cr}^{(1)}$ should either correspond to $n_{cr} = 0$ or to something like $n_{cr} \gg 1$; this can be omitted by changing $\cos\varphi$ in the gap between the two neighboring nodal lines of the waves in zone 1, at the shell edge.[3]

When $M_* = 0$ the second condition ($n_{cr} = 0$) is automatically satisfied and value $\lambda_{cr}^{(1)}$ could be defined as if the only force acting in zone 1 were P.

Where doubt exists as to the fulfillment of the indicated conditions, it can be assumed that Equation [1] provides a lower estimate for λ_{cr} and it can be used only to obtain that estimate. Evidently, because of the following circumstance, we need have no doubts about this. If φ changes substantially inside the strip between the two neighboring nodal lines, then in each such strip D_1 of zone 1, the median absolute value of the axial stress $-P\cos\varphi$ (the basic factor causing stability loss in this zone) is substantially less than its maximum value $|P|$. Similarly, in each such strip D_2 of zone 2, the average shear stress value (the basic factor causing stability loss in zone 2) is substantially less than its maximum value.

[2] With the action of stress P only, its critical value can be achieved for different n and, in particular, when $n = n_{cr} = 0$.

[3] Since the inclination of the nodal lines is small, no discussion is required of the entire strip cut by the two neighboring nodal lines, but only of the gap between them, close to the shell edge, or of some cross section.

Meanwhile, Equation [1] is obtained as a result of replacing, at each point of zones D_1 and D_2, the indicated stresses by their maximum (in modulus) values, which should reduce λ_{cr}. In this case, the values $\lambda_{cr}^{(1)}$ and $\lambda_{cr}^{(2)}$ should be considered as lower estimates of the value λ, in which stability loss can occur correspondingly in zone 1 and in zone 2.

With some restrictions of the considered question, evaluation of shell stability is simplified. Such simplified evaluations are established in what follows and they constitute the basic content of the article.

Section 2 Simplification of Stability Estimates

Let us return to zone 2. As has been indicated, the stability of this zone is considered as if it were a section of a cylindrical shell, loaded by twisting moment $\tilde{M}_*$ and pressure q_*, which would now be considered as internal.

Let internal pressure q_* be sufficiently large so that condition [compare with Eq. [2.13.2] in (2)]

$$-\frac{q_* R^3}{|\tilde{M}_*|} \geq \frac{0.475}{\sqrt[8]{1-\nu^2}} \sqrt[4]{\frac{hR}{l^2}} \qquad [2]$$

is satisfied.

Under pressure and twisting, the angle ψ of nodal line inclination to the shell axis, is determined from equality [(1), p. 107]

$$\tan \psi = (\mu_1 + \mu_2)/2n_{cr}$$

where the values of parameters μ_1 and μ_2 correspond to n_{cr}. Employing the designation M and N, introduced in (2), the preceding equality can be rewritten in the form

$$\tan \psi = \frac{\sqrt{\pi}}{\sqrt[8]{12(1-\nu^2)}} \sqrt[4]{\frac{Rh}{l^2}} \times \frac{M_1 + M_2}{2N_{cr}}$$

Further, as before, in notation of (2) and on the basis of results obtained there, if conditions in Equation [2] are satisfied, one can consider that

$$\tfrac{1}{2}(M_1 + M_2) = M_0 \qquad N_{cr} = N_0$$

and write the equality [(2), pp. 12–14]

$$\frac{M_1 + M_2}{2N_{cr}} = \frac{M_0}{N_0} = \frac{1}{\theta_0} = \frac{23}{\alpha} = \frac{3.52 t_{2*}^0}{3.94 s_*^0} =$$

$$0.7725 \sqrt[8]{1-\nu^2} \sqrt[4]{\frac{l^2}{hR}} \frac{t_{2*}}{s_*}$$

where

$$s_* = \frac{\tilde{M}_*}{2\pi R^2 hE} \qquad t_{2*} = -\frac{q_* R}{hE}$$

Hence, we obtain

$$\tan \psi = \frac{t_{2*}}{s_*}$$

In order to establish the greatest value of ψ, at which Equation [1] can be considered sufficiently exact, a study should be made of the imaginary nodal line intersecting the middle of generatrix $\varphi = \pi/2$ if $M_* \leq 0$ or of generatrix $\varphi = -(\pi/2)$ if $M_* \geq 0$. By virtue of a certain symmetry, this imaginary nodal line should be, if not the actual line, then the median line for strips, cut by the two actual nodal lines neighboring them.

We use $\Delta\varphi$ to designate the absolute value of the increase in angle φ in the root section, on transition from the end of the indicated nodal line to the end of generatrix $\varphi = \pm(\pi/2)$. Evidently

$$|\tan \psi| = (2R\Delta\varphi)/l$$

$\Delta\varphi = 0.7$ can be taken as the greatest value of $\Delta\varphi$, at which it is possible to ignore any change in φ along the nodal lines under consideration, i.e., to consider that along this line $|\sin \varphi| \approx \sin(\pi/2) = 1$. We would then have the following restriction

$$|\tan \psi| = \left|\frac{t_{2*}}{s_*}\right| \leq 1.4 \frac{R}{l}$$

or if s_* and t_{2*} are expressed by M_* and q_*

$$-\frac{q_* R^2 l}{|\tilde{M}|} \leq \frac{0.7}{\pi} = 0.22 \qquad [3]$$

If conditions in Equation [2] are satisfied, we have (see (2), Eq. [2.9])

$$\lambda_{cr}^{(2)} = 7.3\pi \frac{Eh^2 R^4 |q_*|}{\sqrt{1-\nu^2}\tilde{M}_*^2} \qquad [4]$$

while, if Equation [3] is satisfied, this magnitude would sufficiently accurately characterize the shell stability in zone 2. The conditions in Equations [2 and 3] can be combined in

$$\frac{0.475}{\sqrt[8]{1-\nu^2}} \sqrt{\frac{l}{R}} \sqrt[4]{\frac{h}{R}} \leq \frac{|q_*| R^2 l}{|\tilde{M}|} \leq 0.22 \qquad [5]$$

where pressure q_* is considered to be internal.

Thus, if conditions in Equation [5] are satisfied, the magnitude $\lambda_{cr}^{(2)}$, found from Equation [4], sufficiently accurately characterizes the shell stability in zone 2.

Case of $M_* = 0$

Let $M_* = 0$. In this case, as was indicated in the foregoing, $\lambda_{cr}^{(1)}$ can be determined by considering that only stresses $P = -\lambda Q_* l/\pi R^2$ act in zone 1 (according to the "linear theory" the role of internal pressure in zone 1, in this case, is negligible). Since (see (1), Eq. [5.42′])

$$-\frac{P_{cr}}{h} = \frac{Eh}{k_1 R\sqrt{3(1-\nu^2)}} \qquad [*]$$

then

$$\lambda_{cr}^{(1)} = \frac{\pi E R h^2}{k_1 Q_* l \sqrt{3(1-\nu^2)}} \qquad [6]$$

Here k_1 is a correction factor introduced to the formula for critical stress, obtained according to the linear theory where there is only axial compression.

Theoretically, with an internal pressure $q(q < 0)$, exceeding some comparatively small value, namely when $|q| \geq 0.169\, Eh^2/R^2$, the value P_{cr} can be determined according to Equation [*], when $k_1 = 1$ (this result was obtained in Eq. [5], using the nonlinear theory). Evidently, because of deviations in the original form of the shell and in experimental conditions from the ideal [see (6), Fig. 13], a P_{cr} is obtained which is somewhat less than theoretical. Judging from the results of new experiments conducted on shells which have been manufactured from sheet material [see (6), Fig. 12], it can be considered that the actual value of P_{cr} would equal P_{cr}, according to Equation [*], when $k_1 = 1.5$ if $|q| \geq 0.4\, Eh^2/R^2$.

Below, for generality, coefficient k_1 is kept in a letter form. But, on the basis of what has been said in the results reported in the following, it can theoretically (for ideal shells) be considered that $k_1 = 1$ when $|q| \geq 0.169\, Eh^2/R^2$ and practically assumed that $k_1 = 1.5$, when $|q| \geq 0.4\, Eh^2/R^2$.

Equation [4] and condition [5], when $M_* = 0$, are converted to the following

$$\lambda_{cr}^{(2)} = 1.825\pi \frac{Eh^2 R^2 |q_*|}{\sqrt{1-\nu^2}\ Q_*^2} \qquad [7]$$

$$\frac{0.95}{\sqrt[8]{1-\nu^2}}\sqrt{\frac{l}{R}}\sqrt[4]{\frac{h}{R}} \le \frac{|q_*|\,Rl}{Q_*} \le 0.44 \qquad [8]$$

Consequently, when $M_* = 0$ and condition [8] is satisfied, λ_{cr} is determined according to Equation [1], where $\lambda_{cr}^{(1)}$ and $\lambda_{cr}^{(2)}$ are found from formulas [6 and 7], i.e.

$$\lambda_{cr} = 1.825\,\frac{\pi Eh^2R}{\sqrt{1-\nu^2}\,lQ_*}\min\left(\frac{0.316}{k_1},\ \frac{Rl|q_*|}{Q_*}\right) \qquad [9]$$

If, moreover

$$\frac{Rl|q_*|}{Q_*} < \frac{0.316}{k_1} \qquad [10]$$

then $\lambda_{cr} = \lambda_{cr}^{(2)}$ and a stability loss should result in zone 2 from shear stresses. If, instead of [10], the reverse inequality is correct then $\lambda_{cr} = \lambda_{cr}^{(1)}$, and a stability loss should occur in zone 1.

We now make two observations. First, we note that the right part of Equation [8] is automatically satisfied on satisfaction of conditions in [10]. Secondly, it was already indicated that the evaluations of stability in zone 2 as a section of a cylindrical shell, loaded by pressure λq_* and moment $\lambda\tilde{M}_*$ can always (independently of the angle of inclination of nodal lines) serve as the lower evaluation of the stability in this zone, i.e., it can be considered that when $\lambda < \lambda_{cr}^{(2)}$ zone 2 loses no stability. Further, independently of the satisfaction of conditions in Equation [2], the actual value of $\lambda_{cr}^{(2)}$ (i.e., the critical value λ for shells, loaded by pressure λq_* and moment $\lambda\tilde{M}_*$) would be no less than the value of $\lambda_{cr}^{(2)}$, from Equation [4]. This statement was based on that circumstance, shown in [(2), p. 147], that curve Λ (according to which the actual $\lambda_{cr}^{(2)}$ can be found) was located everywhere no lower than the dotted parabola shown on this figure, which corresponds to Equation [4] (the indicated curves are shown in Fig. 4).

Thus, according to Equation [4] value $\lambda_{cr}^{(2)}$ can always be considered as the lower estimate of the stability in zone 2, i.e., when $\lambda < \lambda_{cr}^{(2)}$, where $\lambda_{cr}^{(2)}$ is determined from Equation [4], zone 2 loses no stability.

Having in mind the remarks which have been made, it is possible to report on the results preceding them in the following way.

When transverse force $Q = \lambda Q_*$ and internal pressure $q = \lambda q_*$ act on the shell, and $M_* = 0$, we have:

1 On satisfaction of the conditions in Equation [8], λ_{cr} is determined from Equation [9]. Independently of the satisfaction of conditions in [8], Equation [9] can in any case serve as the lower estimate of magnitude λ_{cr}.

2 On satisfaction of condition

$$\frac{0.95}{\sqrt[8]{1-\nu^2}}\sqrt{\frac{l}{R}}\sqrt[4]{\frac{h}{R}} \le \frac{|q_*|Rl}{Q_*} < \frac{0.316}{k_1} \qquad [11]$$

the magnitude λ_{cr} equals the value $\lambda_{cr}^{(2)}$ according to Equation [7], and a stability loss occurs in zone 2 from the shear stresses.

3 If condition

$$\frac{|q_*|Rl}{Q^*} < \frac{0.316}{k_1} \qquad [12]$$

is satisfied, magnitude λ_{cr} equals value $\lambda_{cr}^{(1)}$ according to Equation [6], and a stability loss takes place in zone 1 from compressive stresses.

Case of $M_* \neq 0$

Now, let $M_* \neq 0$. Since $|\tilde{M}_*| = 2RQ_* + |M_*|$ then condition [5] can be written in the form

$$\frac{0.95}{\sqrt[8]{1-\nu^2}}\sqrt{\frac{l}{R}}\sqrt[4]{\frac{h}{R}} \le \frac{|q_*|Rl}{Q_* + (1/2R)|M_*|} \le 0.44 \qquad [13]$$

If this condition is satisfied, Equation [4] holds, i.e.

$$\lambda_{cr}^{(2)} = 1.825\,\frac{\pi}{\sqrt{1-\nu^2}}\,\frac{Eh^2R^2|q_*|}{[Q_* + (1/2R)|M_*|]^2} \qquad [14]$$

Thus, if condition [13] is satisfied, then λ_{cr} is determined from Equation [1], where $\lambda_{cr}^{(2)}$ can be found from Equation [14], and $\lambda_{cr}^{(1)}$ graphically. Independently of the satisfaction of condition [13], λ_{cr}, found in this way, can always be considered (on the basis of the considerations reported in the foregoing) as the lower estimate of λ_{cr}.

We note that the value of $\lambda_{cr}^{(1)}$, when $M_* \neq 0$, is evidently no greater than its value when $M_* = 0$, which is determined from Equation [6]. Further, the value $\lambda_{cr}^{(2)}$, from Equation [14], can always be considered as the lower estimate of magnitude $\lambda_{cr}^{(2)}$, i.e., when $\lambda < \lambda_{cr}^{(2)}$, where $\lambda_{cr}^{(2)}$ is determined from Equation [14], no stability loss occurs in zone 2. Keeping these remarks in mind and comparing formulas [6 and 14], the following can be asserted: When $M_* \neq 0$ a sufficient condition of shell stability loss in zone 1 is the inequality

$$\frac{\pi ERh^2}{k_1Q_*l\sqrt{3(1-\nu^2)}} < \frac{1.825\pi}{\sqrt{1-\nu^2}} \times \frac{Eh^2R^2|q^*|}{(Q_* + (1/2R)|M_*|)^2}$$

or (after the evident simplifications)

$$\frac{|q_*|Rl}{Q_*[1 + (|M_*|/2RQ_*)]^2} > \frac{0.316}{k_1} \qquad [15]$$

It is evident that if, when $M_* = 0$, zone 2 is less stable than zone 1, then the same would be true even when adding M_* (of the zones of two generatrices $\varphi = \pm(\pi/2)$, each of which could be taken for zone 2, we take that one which is zone 2 when moment M_* is added). If we proceed from this position, then it can be asserted that bulging takes place in zone 2, if $\lambda_{cr}^{(2)} < \lambda_{cr}^{(1)}$, where $\lambda_{cr}^{(1)}$ and $\lambda_{cr}^{(2)}$ were computed, when $M_* = 0$. But that would be the case if condition [11] were satisfied, and then $\lambda_{cr} = \lambda_{cr}^{(2)}$, where $\lambda_{cr}^{(2)}$ would already have been computed with the calculation of M_*.

The value of $\lambda_{cr}^{(2)}$ could be found from Equation [14] if condition [13] were satisfied. Taking all this into consideration and noting that the right-hand part of condition [13] is automatically satisfied when satisfying the right-hand part of condition [11] ($k_1 \geq 1$) we come to the following conclusion.

If conditions

$$\frac{0.95}{\sqrt[8]{1-\nu^2}}\sqrt{\frac{l}{R}}\sqrt[4]{\frac{h}{R}} \le \frac{|q_*|Rl}{Q_* + (1/2R)|M_*|}$$

$$\frac{|q_*|Rl}{Q_*} < \frac{0.316}{k_1}$$

..... [16]

are satisfied, we have $\lambda_{cr} = \lambda_{cr}^{(2)}$ where $\lambda_{cr}^{(2)}$ is determined from Equation [14] and a stability loss takes place in zone 2.

Thus, where transverse force $Q = \lambda Q_*$, internal pressure $q = \lambda q_*$ and twisting moment $M = \lambda M_* (M_* \neq 0)$ all act on a shell, then:

1 Under Equation [15], λ_{cr} equals $\lambda_{cr}^{(1)}$ (can be graphically determined) and a stability loss occurs in zone 1.

2 Under conditions [16] λ_{cr} equals $\lambda_{cr}^{(2)}$, which is determined from Equation [14], and a stability loss takes place in zone 2.

Section 3 Simplified Estimates of Shell Stability Loaded by Transverse and Fixed Values of Twisting Moment and Internal Pressure

It was assumed in the foregoing that when loading there is a simultaneous and identical change of all component loads (transverse force, twisting moment and pressure), i.e., that $Q = \lambda Q_*$, $M = \lambda M_*$ and $q = \lambda q_*$.

Now, let there be only a transverse force load $Q = \lambda Q_*$, with fixed (during loading) values of $M = M_*$ and $q = q_*$, while pressure q_* is internal.

In this case, the simplified estimates of critical force Q_{cr} and the conditions of their applicability can be easily obtained from the previous results.

Case of $M_* = 0$

The transition from the case when $Q = \lambda Q_*$ and $q = \lambda q_*$ to the case considered now, when $Q = \lambda Q_*$ and $q = q_* = \text{const}$, can be effected in the following manner.

Let us introduce quantities $Q_{cr} = \lambda_{cr} Q_*$ and $q_{cr} = \lambda_{cr} q_*$, instead of Q_* and q_*, into Equations [6, 7 and 9] and the conditions of their applicability, assuming, especially that $Q_*/|q_*| = \lambda_{cr} Q_*/|\lambda_{cr} q_*| = Q_{cr}/|q_{cr}|$. Then, we replace q_{cr} by q. Then, instead of the results, reported in foregoing material [see also (1, 2 and 3)] and related to the case of simple loading, we find that when loading with a transverse force $Q = \lambda Q_*$, with a fixed internal pressure q and the absence of a twisting moment, the following holds:

1 Value Q_{cr} is determined from the formula (see Eq. [9])

$$Q_{cr} = \min \frac{2.4hRE}{\sqrt[4]{1-\nu^2}} \left(\frac{1.758}{k_1 \sqrt[4]{1-\nu^2}} \times \frac{h}{l}, \sqrt{\frac{|q|}{E}} \right) \quad [17]$$

if, for this value of Q_{cr}, condition (see Eq. [8]) is satisfied

$$\frac{0.95}{\sqrt[4]{1-\nu^2}} \sqrt{\frac{l}{R}} \sqrt[4]{\frac{h}{R}} \leq \frac{|q|Rl}{Q_{cr}} \leq 0.44 \quad [18]$$

Independently of the satisfaction of condition [18], Equation [17] can serve, in any case, as the lower estimate of Q_{cr}.

2 Value Q_{cr} is determined from the formula (see Eq. [7])

$$Q_{cr} = \frac{2.4}{\sqrt[4]{1-\nu^2}} hR \sqrt{E|q|} \quad [19]$$

if (see Eq. [11])

$$\frac{0.95}{\sqrt[4]{1-\nu^2}} \sqrt{\frac{l}{R}} \sqrt[4]{\frac{h}{R}} \leq \frac{\sqrt[4]{1-\nu^2}}{2.4} \times \frac{l}{h} \sqrt{\frac{|q|}{E}} < \frac{0.316}{k_1} \quad [20]$$

(in this case, the shell's stability loss occurs in zone 2 from shear stresses).

3 Value Q_{cr} is determined from formula (see Eq. [6])

$$Q_{cr} = \frac{\pi ERh^2}{k_1 l \sqrt{3(1-\nu^2)}} \quad [21]$$

if (see Eq. [12])

$$\frac{|q|}{E} > \frac{0.316\pi}{\sqrt{3(1-\nu^2)} k_1^2} \times \frac{h^2}{l^2} \quad [22]$$

(in this case, the shell's stability loss occurs in zone 1 from compressive forces).

Case of $M_* \neq 0$

Let us consider that case where the shell is loaded by transverse force Q with constant values for the twisting moment $M = \text{const} \neq 0$ and the internal pressure $q = \text{const}$.

Keeping in mind the evaluation of the stability in zone 1, where force Q causes only axial stress $P = -Ql/\pi R^2$, we designate by P_{cr} the critical value of the compressive axial force, which uniformly loads the shell with constants M and q. The value P_{cr} can be found graphically; this was reported in (1 and 2) for the simultaneously changing load components.

In the given case, the graphical method of determining P_{cr} consists of the following. With different fixed natural $n > 1$ in the upper semiplane (μ, y), curves $y = y(\mu)$ are plotted according to the formula

$$y = \epsilon \frac{(\mu^2 + n^2)^2}{\mu^2} + \frac{u^2}{(\mu^2 + n^2)^2} - 2s_* \frac{n}{\mu} + t_{2*} \frac{n^2}{\mu^2}$$

where

$$s_* = M/2\pi R^2 hE$$
$$t_{2*} = -qR/hE$$

and horizontal chords are put in them with lengths of $2m\pi R/l$ $(m = 1, 2, \ldots)$. The ordinate of the lowest of these chords for all of indicated family of $y = y(\mu)$ curves is $k_1 P_{cr}/Eh$.

Now, as was done in the previous case of $M_* = 0$ (see foregoing material), one can come to the following conclusions for the case under consideration (loading the shell by transverse force Q with fixed M and q).

1 $Q_{cr} = -\pi R^2 P_{cr} l$, and stability loss takes place in zone 1, if this value of Q_{cr} satisfies condition [see (15)]

$$\frac{|q|Rl}{Q_{cr}[1 + (|M|/2RQ_{cr})]^2} > \frac{0.316}{k_1}$$

2

$$Q_{cr} = \frac{2.4}{\sqrt[4]{1-\nu^2}} hR \sqrt{E|q|} - \frac{1}{2R}|M|$$

(see Eq. [14]) and a stability loss takes place in zone 2 if this value of Q_{cr} satisfies conditions (see Eq. [16])

$$\frac{0.95}{\sqrt[4]{1-\nu^2}} \sqrt{\frac{l}{R}} \sqrt[4]{\frac{h}{R}} \leq \frac{|q|Rl}{Q_{cr} + (1/2R)|M|} \qquad \frac{|q|Rl}{Q_{cr}} < \frac{0.316}{k_1}$$

Section 4 Approximate Evaluation of Shell Stability When, Among the Load Components, There Is a Transverse Bending Force, Uniformly Distributed Along the Entire Shell

As a supplement to all that has been explained in the foregoing, let us consider that case when, among the load components, there is a transverse bending force Q_*', uniformly distributed along the entire shell (Q_*' is the total magnitude of this force; it is considered to be directed as force Q, and positive in this direction).

The results, reported in (1) (Section 33), make possible an approximation of shell stability when only force $Q' = \lambda Q'_*$ $(\lambda > 0)$ operates. The explanation goes as follows. It is considered that shell bulging can occur in any of the zones: 1, 2 or 3. Zones 1 and 2 were defined in the foregoing, while generatrix zone $\varphi = 0$ is considered as zone 3. The stability loss in zone 1 is considered as if this zone were a section of the shell, loaded by axial compression stress $P' = \lambda P_*' = -\lambda Q_*' l/2\pi R^2$ and by internal pressure $q' = \lambda q_*' = -\lambda Q_*'/2\pi Rl$.

The stability loss in zone 2 is considered as if this zone were a section of the shell, loaded by twisting moment $\tilde{M}' = \lambda \tilde{M}_*' = \pm \lambda l, 4RQ_*'$ (the selection of signs is the same as for M; see foregoing material). Finally, in evaluating shell stability from above and below in zone 3, this zone is considered as a section of the shell, loaded correspondingly by external pressure $q' = \lambda q_*' = \lambda Q_*'/2\pi Rl$ (for a lower estimate) and by the indicated pressure together with the axial tensile stress $P' = \lambda P_*' = \lambda Q_*' l/2\pi R^2$ (for an upper estimate). But such an upper estimate of the stability of zone 3 should differ little from the indicated lower estimate. In fact, corresponding to tensile stress P_*' is the dimensionless normal stress (stress, divided by E)

$$t_{1*} = P_*'/Eh = Q_*' l/2\pi EhR^2$$

and to external pressure q' dimensionless normal stress

$$t_{2*} = -q_*' R/Eh = -Q_*' R/2\pi EhRl$$

The critical values of compressive dimensionless stresses in an axial (t_{1cr}) and circumferential (t_{2cr}) direction are deter-

mined by the formulas

$$t_{1cr} = -\frac{h}{k_1 R\sqrt{3(1-\nu^2)}}$$

$$t_{2cr} = -\frac{2\pi}{3\sqrt{6}(1-\nu^2)^{3/4}} \times \frac{h}{l}\sqrt{\frac{h}{R}}$$

Hence, we obtain

$$t_{1*}{}^0 = t_{1*}/|t_{1cr}| = \frac{k_1 Q_*{}' l\sqrt{3(1-\nu^2)}}{2\pi E R h^2}$$

$$t_{2*}{}^0 = t_{2*}/|t_{2cr}| = -\frac{3\sqrt{6}(1-\nu^2)^{3/4} Q_*{}'}{4\pi^2 E h^2}\sqrt{\frac{R}{h}}$$

and consequently

$$\frac{t_{1*}{}^0}{|t_{2*}{}^0|} = \frac{\sqrt{2}\,\pi k_1}{3(1-\nu^2)^{3/4}} \times \frac{l}{R}\sqrt{\frac{h}{R}} = \frac{2}{3}\sqrt[4]{3\pi^2 k_1}\sqrt{\epsilon^{1/2}(l/\pi R)^2} \qquad [23]$$

where

$$\epsilon = h^2/12(1-\nu^2)R^2$$

For shells of medium length, which are being considered here, we have

$$\epsilon^{1/2}(l/\pi R)^2 \ll 1$$

[see (2), p. 138]. In particular, there is extensive use of shells for which condition [1′] of (2) is satisfactory

$$1000\epsilon^{1/2} \leq (\pi R/l)^2 \leq 0.02\epsilon^{-1/2} \qquad [24]$$

hence

$$\epsilon^{1/2}(l/\pi R)^2 \leq 0.001 \qquad [25]$$

Therefore, from Equation [23], we obtain $t_{1*}{}^0/t_{2*}{}^0 \ll 1$, while, if Equation [25] is kept in mind then, in any case, $t_{1*}{}^0/t_{2*}{}^0 \leq 0.274k_1$.

But, from Fig. 2 of (7), where it is considered that $k_1 = 1$, it is evident that with the simultaneous action of external pressure and axial elongation, the influence of the latter is negligible if the $t_1{}^0/t_2{}^0$ ratio is considerably less than a unit; this also justifies the statement made in the foregoing [it should be considered that $t_{1*}{}^0/t_{2*}{}^0 \geq 0.274$ when using Fig. 2 of (7)].

Thus, when only force $Q' = \lambda Q_*{}'$ acts on the shell, stability loss in zone 3 can be considered as if this zone were a section of the shell, loaded only by external pressure $q' = \lambda q_*{}' = \lambda Q_*{}'/2\pi Rl$. Consequently, with the stability loss in zone 3, the critical value of parameter $\lambda(\lambda_{cr}{}^{(3)})$ would be $\lambda_{cr}{}^{(3)} = q_{cr}/q_*{}'$ where critical pressure $q_{cr} = -t_{2cr}(Eh/R)$, i.e.

$$\lambda_{cr}{}^{(3)} = 5.37\frac{Eh^2}{(1-\nu^2)^{3/4}Q_*{}'}\sqrt{\frac{h}{R}} \qquad [26]$$

The critical value of parameter $\lambda(\lambda_{cr}{}^{(2)})$, corresponding to the stability loss in zone 2, is determined from equality $|\tilde{M}'| = |M_{cr}|$, where M_{cr} is the critical value of the twisting moment for the examined shell (see, for example, (2), Eq. [1.9]), so that

$$\lambda_{cr}{}^{(2)} = 3.32\frac{Eh^2}{(1-\nu^2)^{5/8}Q_*{}'}\sqrt[4]{\frac{hR}{l^2}} \qquad [27]$$

As was indicated earlier, in conformity with the "linear" theory, internal pressure does not influence the stability of cylindrical shells, when axial compression exists. According to the "nonlinear" theory, the internal pressure ceases to exert a compensating action, beginning with the small value $|q| > 0.169Eh^2/R^2$ and for such q, the linear theory was correct without calculating q. Therefore, critical value $\lambda(\lambda_{cr}{}^{(1)})$, corresponding to the stability loss in zone 1, is obtained from equality $P' = P_{cr}$, where P_{cr} is the critical value of the axial force when only axial compression exists (see Eq. [*]). Thus, we obtain

$$\lambda_{cr}{}^{(1)} = 3.62\frac{Eh^2}{k_1\sqrt{3(1-\nu^2)}Q_*{}'} \times \frac{R}{l} \qquad [28]$$

For shells, satisfying condition [24], we have (from the left-hand part of this condition)

$$\frac{h}{R} \leq \sqrt{12(1-\nu^2)}\,\frac{\pi^2}{1000} \times \frac{R^2}{l^2}$$

hence, it follows that

$$\sqrt{\frac{h}{R}} \leq 0.185\sqrt[4]{1-\nu^2}\,\frac{R}{l} = 0.185\sqrt[4]{1-\nu^2}\sqrt{\frac{R}{h}}\sqrt{\frac{Rh}{l^2}} \qquad [29]$$

and

$$\sqrt{\frac{h}{R}} \leq 0.43\sqrt[8]{1-\nu^2}\sqrt[4]{\frac{Rh}{l^2}} \qquad [30]$$

It is evident from Equations [26–28] and inequalities [29 and 30] that $\lambda_{cr}{}^{(3)} < \lambda_{cr}{}^{(2)}$ and $\lambda_{cr}{}^{(3)} < \lambda_{cr}{}^{(1)}$ (considering that $k_1 \geq 3, 6$).

Thus, the cylindrical shell, found under the action of only force $Q' = \lambda Q_*{}'$ and satisfying condition [24], should (according to the approximate evaluation) bulge in zone 3, and the critical value λ for it should equal $\lambda_{cr}{}^{(1)}$, according to Equation [28].

Let us now return to the case when forces $Q' = \lambda Q_*{}'$ act on the shell in conjunction with the loads considered in the foregoing: $Q = \lambda Q_*$, $M = \lambda M_*$ and $q = \lambda q_*$ ($q_* < 0$, i.e., internal pressure). In this case, the approximate evaluation of shell stability consists (if the foregoing explanation is kept in mind) in isolated considerations of zones 1, 2 and 3 as shell sections, loaded correspondingly by the following loads.

Loads in zone 1: Compressive axial stress $P'' = \lambda P_*{}''$, twisting moment $M = \lambda M_*$ and internal pressure $q'' = \lambda q_*{}''$ where

$$P_*{}'' = P_* + P_*{}' = -\frac{l}{\pi R^2}\left(Q_* + \frac{1}{2}Q_*{}'\right)$$

$$q_*{}'' = q_* + q_*{}' = q_* - Q_*{}'/2\pi Rl \qquad [31]$$

Loads in zone 2: Twisting moment $\tilde{M}'' = \lambda\tilde{M}_*{}''$ and internal pressure $q = \lambda q_*$, where

$$\tilde{M}_*{}'' = \tilde{M}_* + \tilde{M}_*{}' = \mp[2R(Q_* + 0.7Q_*{}') + |M_*|] \qquad [32]$$

Loads in zone 3: Pressure $q'' = \lambda q_*{}''$ and twisting moment $M = \lambda M_*$ (for a lower stability estimate) with additions of tensile axial stress $P'' = \lambda P_*{}''$ (for an upper stability estimate), where

$$q_*{}'' = q_* + q_*{}' = q_* + Q_*{}'/2\pi Rl$$

$$P_*{}'' = P_* + P_*{}' = \frac{l}{\pi R^2}\left(Q_* + \frac{1}{2}Q_*{}'\right) \qquad [33]$$

Comparing the load in zone 1, for this case, with the load in this zone for the case considered in Sections 2 and 3, we observe that these loads are qualitatively equal (see foregoing material). Therefore, evaluation of the stability in zone 1 of the case under consideration can be effected in the manner indicated in Sections 2 and 3. However, there should be a replacement, in the employed results and the conditions of their applicability, of Q_* and q_* by the quantities (see Eq. [31])

$$Q_*{}'' = Q_* + \tfrac{1}{2}Q_*{}' \text{ and } q_*{}'' = q_* - Q_*{}'/2\pi Rl$$

In particular, when $M_* = 0$, internal pressure does not have to be considered; the appropriate critical value of $\lambda(\lambda_{cr}^{(1)})$ can be determined from Equation [6] with the replacement of Q_* by Q_*''. An analogous situation occurs in relation to zone 2 as well. Therefore, it is possible to evaluate the stability of zone 2 as indicated in Sections 2 and 3. However, in the employed results and the conditions of their applicability, the magnitude Q_* should be replaced by magnitude (see Eq. [32])

$$Q_*'' = Q_* + 0.7Q_*'$$

As a result, the appropriate critical value of parameter $\lambda(\lambda_{cr}^{(2)})$ will be found.

Of course, an estimate of the stability of zone 3 should be made only when $q_*'' > 0$ i.e., when $Q_*'2\pi Rl > |q_*|$. In this case, it is easy to obtain the lower stability value with the aid of curve Λ, shown in Fig. 4. In addition, quantities t_2^0 and s^0, introduced in (2) would be

$$t^0 = \lambda t_{2*}{}^0 = \lambda t_{2*}/|t_{2cr}| = -\lambda q_*''R/Eh|t_{2cr}| = -\lambda \frac{3\sqrt{6}}{2\pi}(1-\nu^2)^{3/4}\frac{Rl}{h^2}\sqrt{\frac{R}{h}} \times \frac{q_*''}{E} \quad [34]$$

$$s^0 = \lambda s_*{}^0 = \lambda M_*/2\pi R^2 h E s_{cr} = \lambda \frac{(1-\nu^2)^{5/8}}{1.48\pi}\sqrt[4]{\frac{l^2}{hR}} \times \frac{M_*}{ERh^2} \quad [35]$$

where $s_{cr} = 0.74(1-\nu^2)^{-5/8}(h, R)\sqrt[4]{(hR/l^2)}$ is a critical dimensionless shear stress (stress, divided by E) with the pure twisting of medium length cylindrical shells.

The lower estimate of the critical value of λ, for zone 3, $(\lambda_{cr}^{(3)})$, is derived with the aid of Λ in the following manner. On the coordinate plane, where curve Λ was plotted, point A_* is plotted with coordinates $t_{2*}{}^0$ and $s_*{}^0$, determined from equalities [34 and 35]. A ray is produced through the origin of coordinates O and point A_* until it intersects curve Λ at some point A. The ratio of segments OA/OA_* is $\lambda_{cr}^{(3)}$.

The upper estimate of the critical value of λ in zone 3, $(\bar{\lambda}_{cr}^{(3)})$ is determined as the critical value of λ when there is a simultaneous action on the whole cylindrical shell of pressure $q'' = \lambda q_*''$, twisting moment $M = \lambda M_*$ and axial stress $P'' = \lambda P_*''$, with known q_*'', M_*, P_*'', as indicated in the foregoing. Determination of such a critical value of λ can be effected graphically as indicated in (1) (see Section 30).

For the approximate evaluation of the stability of zone 3, i.e., with an approximate determination of the appropriate critical value of λ, which we designate by $\lambda_{cr}^{(3)}$, it can be considered that

$$\lambda_{cr}^{(3)} = \tfrac{1}{2}(\underline{\lambda}_{cr}^{(3)} + \bar{\lambda}_{cr}^{(3)})$$

In order to explain the advisability of the upper estimate, it should be established, using the curves shown in (7) in Figs. 1 and 2, to what extent the tensile stress P'' can increase the critical value of external pressure q'' and twisting moment M where they act separately. Where this increase is slight it is not advisable to determine $\lambda_{cr}^{(3)}$ (evidently, in this case it should differ little from $\lambda_{cr}^{(3)}$) and it can be assumed that

$$\lambda_{cr}^{(3)} = \underline{\lambda}_{cr}^{(3)}$$

Having evaluated the stability of zones 1, 2 and 3, i.e., having defined $\lambda_{cr}^{(1)}$, $\lambda_{cr}^{(2)}$ and $\lambda_{cr}^{(3)}$, we obtain the critical value $\lambda(\lambda_{cr})$ for the shell as a whole as

$$\lambda_{cr} = \min(\lambda_{cr}^{(1)}, \lambda_{cr}^{(2)}, \lambda_{cr}^{(3)})$$

Section 5 Experimental Study

Under the direction of the author, R. I. Kshnyakin tested 33 Duraloy shells for stability, employing the combined action of transverse force Q, twisting moment and internal pressure. The shells were cut along with the flanges from tubes, with sufficiently high accuracy. The longitudinal section of a shell is shown in Fig. 1. Shells have the following rated thicknesses h (in mm): 0.3, 0.4, 0.5 and 0.6. As a rule, deviations in thickness, at different points of the shell, from rated do not exceed ± 5 per cent. Measurements of shell thicknesses were made at three points of each of four generatrices, equidistant from one another (at the flanges and in the center of the shell). In the shell calculations, the least of the average measurement results along each of the indicated

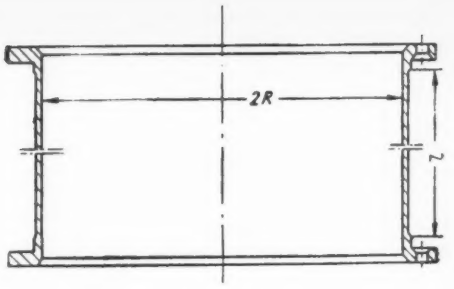

Fig. 1

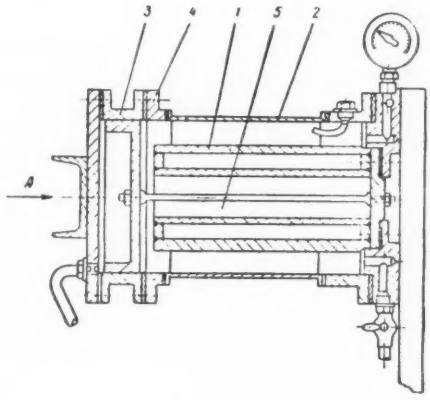

Fig. 2

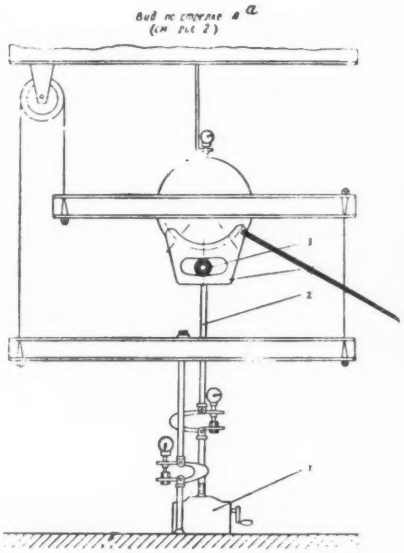

Fig. 3 Translation: [a] View along arrow A (see Fig. 2)

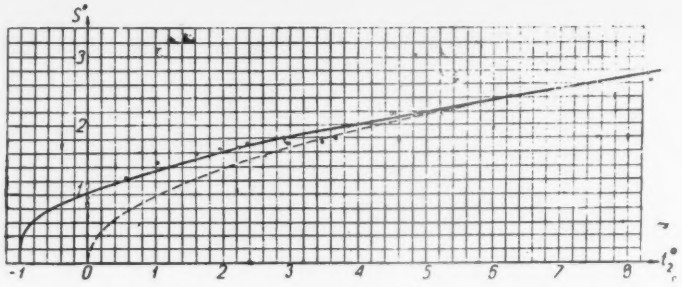

Fig. 4

generatrices were taken for its thickness. All tested shells had an identical diameter $2R = 220$ mm, their lengths are $l = 108$ and $l = 228$ mm.

Tests were made on special four-component devices, whose layout is shown in Fig. 2. Pressure is created by an oil pump, delivering oil to the housing between the casing of the unit 1 and the shell 2. Twisting moment is created with the aid of the system of levers shown in Fig. 3, and is transmitted to one of the two flanges of the shell (the other flange is attached to rigid part of the device and is loaded by a reactive moment). A transverse force is created by jack 1. Stress from the jack is transmitted through connecting rod 2 and bearing 3 to link 4 (Fig. 3), attached by bolts to ring 3 and shell flange 4 (see Fig. 2). Bearing 3 (Fig. 3) serves as the central application of the bending transverse force Q. In order to eliminate the axial tensile stress, created by the oil pressure, a steel plate 5 (see Fig. 2) was introduced into the device. Its action was elastic in the plane of force Q and rigid in the plane perpendicular to the line of action of this force.

Loading was done in small stages so that at the end of each loading stage the adopted relationship between the components remained constant. For all shells, except those with numbers 3, 4, 5, 27 and 33, this relationship was so selected that the theoretical value of $\lambda_{cr}^{(1)}$ and $\lambda_{cr}^{(2)}$ satisfied condition $\lambda_{cr}^{(2)} < \lambda_{cr}^{(1)}$, i.e., so that bulging occurred in zone 2 (see Fig. 2). Shells with numbers 3, 4, 5 and 27 were tested with the action of transverse force Q and internal pressure q, taken so that $\lambda_{cr}^{(1)} < \lambda_{cr}^{(2)}$, i.e., theoretically, they should bulge in zone 1. Finally, shells with number 33, were tested with a Q and q so that $\lambda_{cr}^{(1)} \approx \lambda_{cr}^{(2)}$, i.e., theoretically, this shell should have bulged in the gap between zones 1 and 2 when λ_{cr} was somewhat less than $\lambda_{cr}^{(1)}$ and $\lambda_{cr}^{(2)}$. The moment of stability loss was fixed on the basis of visual observation of the shell surface and of the strips of light reflected on its surface (these strips

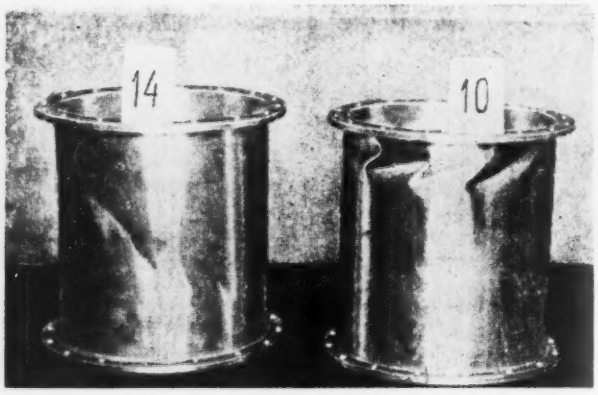

Fig. 5

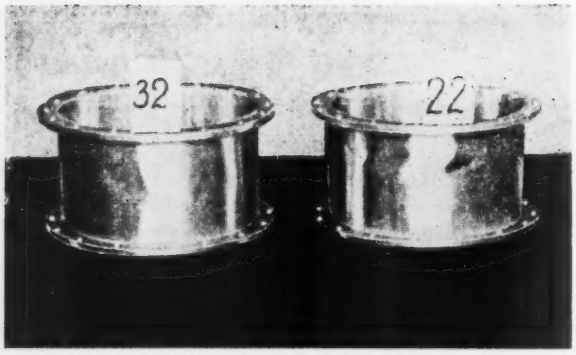

Fig. 6

Fig. 7

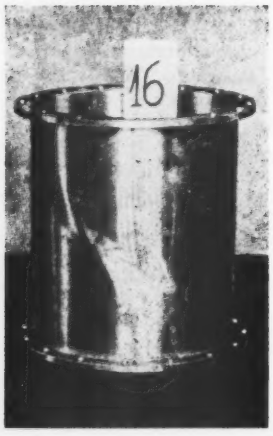

Fig. 8

bend markedly with the slightest bulging); it was corrected on the basis of indicator readings. The indicator measured deflection at shell ends (where there is a stability loss, a destruction takes place of the rectilinear law between the indicated deflection and transverse force).

The bulging of all the shells, except the 3rd, 4th, 5th, 27th and 33rd, took place in zone 2, as was expected. Test results for all shells are shown by the circles in Fig. 4. The two intervals, shown on the ordinate axis, embrace the experimental points which are obtained as a result of tests made on seven shells with no pressure ($q = 0$). The coordinates in Fig. 4 are the quantities t_2^0 and s^0, the significance of which is the following (these quantities have already been discussed in Section 5).

The quantity t_2^0 is the relationship of the circumferential normal stress, acting on the shell, to the critical value of this stress, where there is uniform transverse compression. The quantity s^0 is the relationship of the shear stress, acting on the shell, to the critical value of this stress, where there is pure twisting. The points on curve Λ (the solid line on Fig. 4) determine the theoretical critical values of quantities t_2^0 and s^0, where there is simultaneous action of the stresses corresponding to them. When the load components are $Q = \lambda Q_*$, $M = \lambda M_*$ and $q = \lambda q_*$ and bulging occurs in zone 2, i.e., when $\lambda_{cr} = \lambda_{cr}^{(2)}$, then t_2^0 and s^0 are determined from the pressure $q = \lambda q_*$ and twisting moment $\bar{M} = \lambda \bar{M}_* = \pm\lambda(2RQ_* + |M_*|)$ (see foregoing material). In order to determine the theoretical critical values of t_2^0 and s^0, in this case, it is necessary to plot point A_* with coordinates $t_{2*}{}^0$ and $s_*{}^0$, corresponding to quantities q_* and M_*, on the coordinate plate of Fig. 4; it is also necessary to trace a ray through the origin of coordinates O and point A_* until it intersects with curve Λ at some point A. The OA/OA_* ratio would be equal to the theoretical value $\lambda_{cr} = \lambda_{cr}^{(2)}$.

If a ray were traced in Fig. 4 through any experimental point B_* and the origin of coordinates O to its intersection with curve Λ at point B, then the OB/OB_* ratio would be equal to the ratio of the theoretical value of λ_{cr} to the experimental value of λ_{cr} for shells, corresponding to point B_*. For all experimental points this relationship differs comparatively little from a unity (for a series of points there is almost no difference), i.e., the experimental results conform with the theoretical.

For some of the shells, whose test results are shown in Fig. 4, condition [11] or conditions [16] are satisfied, since it was possible to determine $\lambda_{cr} = \lambda_{cr}^{(2)}$ for them in accordance with Equation [7 or 14].

Figs. 5–8 show the nature of bulging for some of the tested shells. These shells were subjected to bending by transverse force with twisting (nos. 14, 32) or with internal pressure (nos. 10, 22, 27), and also with twisting and internal pressure (no. 16).

Shells with numbers 3, 4, 5 and 27 bulged, as was anticipated, in zone 1. The nature of the 27th shell's bulging is shown in Fig. 7.

For these shells, the ratio of the experimental value of λ_{cr} to the theoretical (equal to $\lambda_{cr}^{(1)}$) was 1.09, 1.18, 1.04 and 0.82.

Bulging of the last shell (33rd) occurred, as was expected in the gap between zones 1 and 2 for λ_{cr}, which is approximately 0.63 of theoretical (from $\lambda_{cr}^{(1)} \approx \lambda_{cr}^{(2)}$). As was shown in foregoing material it can be expected that in this case (an exception from theoretical analysis) the critical value λ would be approximately 0.7 of magnitude $\lambda_{cr}^{(1)} \approx \lambda_{cr}^{(2)}$.

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Stability Region of Certain Nonlinear Control Systems

B. N. SKACHKOV

WE CONSIDER the control system

$$\begin{aligned}\dot{\eta} &= r\eta + n\xi \\ \dot{\xi} &= f(\eta, \xi)\end{aligned} \qquad [1]$$

where $f(\eta, \xi)$ is any continuous function of its arguments, having the property

$$f(\eta, \xi)|_{\sigma=0} = 0 \qquad \sigma f(\eta, \xi)|_{\sigma\neq 0} > 0 \qquad [2]$$

Here $\sigma = p\eta - \xi$, and the numbers r, n and p are real constants, with

$$r < 0 \qquad [3]$$

$$n > 0 \qquad [4]$$

(The case $n < 0$ reduces to the foregoing by substituting $-\eta$ for η.)

The system [1] can be considered as describing, for $n_1 = 0$, the motions of the system

$$\begin{aligned}\dot{\eta}_1 &= r_1\eta_1 + n_1\xi \\ \dot{\eta}_2 &= r_2\eta_2 + n_2\xi \\ \dot{\xi} &= f(\eta_1, \eta_2, \xi)\end{aligned} \qquad [5]$$

on the plane $\eta_1 = 0$.

It is shown in (1)[1] [the uniqueness assumption is not essential in (1)] that the system [5] is stable as a whole, if its parameters satisfy the condition

$$1 + \frac{|n_1 p_1|}{r^1} + \frac{|n_2 p_2|}{r_2} > 0 \qquad [6]$$

(For the system [5] we have $\sigma = p_1\eta_1 + p_2\eta_2 - \xi$.)

From condition [6] it follows that the system [1] is stable as a whole when

$$1 + \frac{|np|}{r} > 0 \qquad [7]$$

In the present article we demonstrate the generality of condition [7], which defines the region of parameters under which the system [1] is stable on the whole. It is shown, specifically, that if an inequality inverse to [7] is satisfied (the case of exact equality is not considered) there always exists a function $f(\eta, \xi)$, satisfying all the conditions imposed in the foregoing, which when substituted in the right half of the second equation of [1] makes this system unstable. In this sense, inequality [7] can be called the necessary condition for stability.

When $p \geq 0$, condition [7] coincides with the Hurwitz condition for the system [1], with

$$f(\eta, \xi) = h\sigma, \; h > 0 \qquad [8]$$

This means that if condition [7] is violated when $p > 0$, then the system [1] is unstable even for a function of the form [8].

Consider the case

$$p < 0 \qquad [9]$$

The inequality inverse to [7] can now be rewritten, in view of Equations [3, 4 and 9], in the form

$$r - np > 0 \qquad [10]$$

Consider the phase plane (see figure) of system [1], and lines in this plane satisfying the equations

$$\begin{aligned} r\eta + n\xi &= 0 \qquad l_1 \\ p\eta - \xi &= 0 \qquad l_2 \end{aligned} \qquad [11]$$

The lines [11] divide the plane into four zones. We number these zones in accordance with the combinations of the signs of $\dot{\eta}$ and $\dot{\xi}$ in the zones, taking Equation [2] into account, in the following manner

$$\text{I}--, \; \text{II}+-, \; \text{III}++, \; \text{IV}-+$$

Along the lines [11], we draw the lines

$$\begin{aligned} (r - \epsilon)\eta + n\xi &= 0 \qquad l_1' \\ (p + \mu)\eta - \xi &= 0 \qquad l_2' \end{aligned} \qquad [12]$$

where ϵ and μ are positive numbers, satisfying the inequality

$$\epsilon + n\mu < r - np \qquad [13]$$

It follows from inequality [10] that such numbers exist.

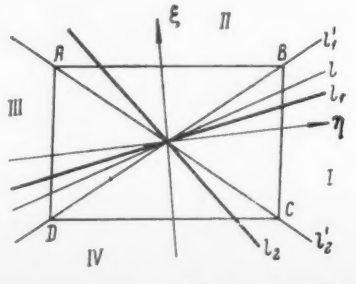
We construct in the phase plane rectangle $ABCD$, which has segments of the lines [12] as its diagonals. It follows readily from [13] that the line adjoining the points A and B of this rectangle intersect the positive half axis η or, what is the same, its slope k is

$$k < 0 \qquad [14]$$

(It is easy to calculate that $\tan^{-1} k = (1/2)\{\tan^{-1}[-(r - \epsilon)/n] + \tan^{-1}(p + \mu)\}$ and consequently $-1 < k < 0$.)

It is clear from the diagram that under our conditions one can readily indicate a set of functions $f(\eta, \xi)$ under which the motions of the system [1] will cross the sides of the rectangle $ABCD$ from the inside out and the system [1] will be unstable. Let us indicate one class of such functions. We draw in the plane η, ξ the line

$$[r - (\epsilon/2)]\eta + n\xi = 0 \qquad l \qquad [15]$$



Translated from *Vestnik Leningradskovo Universiteta* (Bull. of Leningrad State University), no. 1, 1960, pp. 100–103. Translated by J. George Adashko.

[1] Numbers in parentheses indicate References at end of paper.

It is obvious that the slopes of the lines l, l_1 and l_1' satisfy the inequality

$$k_{l_1} < k_l < k_{l_1'}$$

(l can be any line satisfying this inequality.)

The line [15] cuts zones II and IV into parts. The parts adjacent to l_1 will be denoted II_l, IV_l while those adjacent to l_2 will be denoted II_{l2} and IV_{l2}.

Consider the system [1] under the assumption that $f(\eta, \xi)$ satisfies the conditions

$$\begin{gathered} \sigma f(\eta, \xi) \geqslant (1/\beta)\,\sigma^2 \text{ in zones I and III} \\ \sigma f(\eta, \xi) \leqslant \alpha\sigma^2 \text{ in zones } II_{l_2} \text{ and } IV_{l_2} \\ \alpha\sigma^2 \leqslant \sigma f(\eta, \xi) \leqslant (1/\beta)\,\sigma^2 \text{ in zones } II_{l_1} \text{ and } IV_{l_1} \end{gathered} \qquad [16]$$

where

$$0 < \alpha < \frac{kn\epsilon}{r + np - \epsilon} \qquad [17]$$

and

$$0 < \beta < \frac{k\mu}{r + np + n\mu} \qquad [18]$$

(It is easy to prove that under our conditions $(1/\beta) > \alpha$.)

From the continuity condition it follows that $f(\eta, \xi) = \alpha\sigma$ on the line l and $f(\eta, \xi) = (1/\beta)\sigma$ on l_1.

Theorem. The system [1] is unstable if conditions [10 and 16] are satisfied.

Proof. Consider the side AB of rectangle $ABCD$. This side is a segment of a line, the equation of which is

$$-k\eta + \xi = c$$

where c is a certain positive constant. The derivative $-k\dot{\eta} + \dot{\xi}$ of the expression $-k\eta + \xi$, by virtue of system [1], is positive both in zone III and zone II_{l_2}. For zone III this follows from the inequalities [14], $\dot{\eta} > 0$, and $\dot{\xi} > 0$, while for zone II_{l_2}

$$-k\dot{\eta} + \dot{\xi} \geqslant -k\dot{\eta} + c\sigma > -k\dot{\eta} + \frac{kn\epsilon}{r + np - \epsilon}\,\sigma = \frac{-k(r + np)\,[(r - \epsilon)\eta + n\xi]}{r + np - \epsilon} \geqslant 0$$

as follows readily from the inequalities (3, 4, 9, 16 and 17) and $\sigma < 0$.

Thus, $-k\dot{\eta} + \dot{\xi} > 0$ along the entire segment AB. Consequently, motions of the system [1] that intersect AB leave the rectangle $ABCD$. In the same manner, $\dot{\eta} + k\dot{\xi} > 0$ for the side BC, which lies on the line $\eta + k\xi = c$, $c > 0$.

For the part of BC lying in zone II, this follows from the inequalities [14], $\dot{\eta} > 0$, and $\dot{\xi} < 0$. For the part BC, lying in zone I

$$\dot{\eta} + k\dot{\xi} \geqslant \dot{\eta} + \frac{k}{\beta}\,\sigma > \dot{\eta} + \frac{r + np + n\mu}{\mu}\,\sigma = \frac{r + np}{\mu}\,[(p + \mu)\eta - \xi] \geqslant 0$$

as follows from inequalities [16 and 18]. This means that the motions through the side BC also go outside the rectangle. The arguments for sides CD and AD are analogous. This proves the theorem.

Note 1. The example of system [1] demonstrates clearly the difference between systems in which the nonlinear function depends only on $\sigma[\dot{\xi} = \varphi(\sigma)]$ and general systems, in which $\dot{\xi} = f(\eta, \xi)$.

For the system

$$\begin{aligned} \dot{\eta} &= r\eta + n\xi \\ \dot{\xi} &= \varphi(\sigma) \end{aligned}$$

which is obtained from the system [1] by putting $f(\eta, \xi) = \varphi(\sigma)$ the sufficient condition for the stability on the whole, as follows readily, for example from (2), is the inequality

$$r + np < 0$$

This inequality is always satisfied under conditions [3, 4 and 9]. The stability of the system [1] on the whole can be insured, however, only if inequality [7] is satisfied.

Note 2. By a method analogous to the foregoing, it can be proved that inequality [6] is the necessary condition of stability of system [5], also.

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—Original received June 25, 1959

Reviewer's Comment

This paper presents the proof of a necessary condition for stability of a particular class of control systems. The class treated can be considered as one with two variables having cross coupling and including a nonlinear function of the two variables. The nonlinear function controls the rate of one of the variables. This function is defined so that it corresponds to either plus or minus rates depending in what area of the phase plane defined by the two variables the system lies. The dividing line between the two areas is a straight line through the origin of the phase plane. It acts as a switching line. The paper presents a geometric type proof for a necessary condition for stability of the system.

The reviewer has had no contact with treatments of this sort in American control literature. Undoubtedly there are Russian papers using this type of approach since there are references made to such papers, and the Russian literature contains a large amount of material on the theoretical aspects of the stability of nonlinear systems. However, such references are not readily available.

The results of the paper are useful because they are not obvious when the slope of the $\dot{\xi}$ switching line is negative ($p < 0$). Also, they are pertinent because the system defined is reasonably general within its class.

It is felt that the final steps in the proof would be clearer and would indicate the reasoning behind them if they were presented in a reverse fashion. For instance, for the system to leave the rectangle $ABCD$ through the side AB it is correctly stated that $-k\dot{\eta} + \dot{\xi} > 0$. Then, it can be written that

$$-k\dot{\eta} + \alpha\sigma > 0$$

Since σ is negative in the region under consideration

$$\alpha < \frac{k\dot{\eta}}{\sigma} = \frac{k[r + n(\xi/\eta)]}{[p - (\xi/\eta)]}$$

Also, in the regions being considered

$$\frac{\xi}{\eta} > \left\{\frac{r - \epsilon}{n}\right\}$$

Thus

$$\alpha < \frac{k\{r + n[-(r + \epsilon)/n]\}}{p - [-(r + \epsilon)/n]} = \frac{kn\epsilon}{np + r - \epsilon}$$

The same treatment can be applied to the other sides of the rectangle. Otherwise the proof is straightforward.

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Reflection of a Ring Current by a Plasma

V. N. TSYTOVICH

THE question of the forces acting between a medium, occupying a half-space, and a ring current, the plane of which is parallel to the boundary surface, has been considered in (1).[1] The present paper is concerned with some questions of the dynamics of a collision between a ring current and a plane boundary formed by a plasma, the initial velocity of the ring current being in the nonrelativistic range. The main problem then is to find the conditions under which the ring current is elastically reflected from the plasma. In these circumstances it is necessary to distinguish between fast and slow collisions. Fast collisions are those for which the collision time is less than the characteristic time linked with the dissipative processes (plasma conductivity, resistance of ring, etc.). Slow collisions satisfy the opposite condition. However the single condition that the collision process be a fast one is not enough for reflection. It is also necessary that the height of the potential barrier between the ring and the plasma exceed the initial energy of the progressive motion of the ring towards the plasma.

In material which follows it will be shown that, provided definite conditions are fulfilled, even a fairly tenuous plasma may serve as an excellent reflector of such ring currents. The concrete ring current of radius a, considered in the following, is not essential to this general conclusion.

As far as the stability of such a ring current is concerned, it is important to know how much the radius of the ring changes during the conclusion. In the following we derive a criterion, the fulfillment of which corresponds to a negligibly small variation in the radius during the collision.

Let us consider the question of the height of the potential barrier. For fast collisions it is possible to reckon that the condition of conservation of magnetic flux across the ring is sufficiently well fulfilled. If the boundary surface were the boundary of an ideal conductor, then both the ring current and the potential energy would tend to infinity as the ring approached the boundary surface. Actually the potential energy V is then determined from the expression

$$V = \frac{\Phi^2}{2(L - M)}$$

where

L = instantaneous value of self-inductance of loop
M = instantaneous value of mutual inductance with image
Φ = magnetic flux across loop

As the loop approaches the boundary surface L tends to M, while Φ remains constant.

As is easily seen, dispersion properties in the plasma should lead to a limitation of the increase in the potential curve. The potential maximum may be evaluated by quite general means, without resorting to detailed calculations.

It is easy to understand that in the presence of dispersion the image ought to spread in the direction of motion of the original loop, because to each harmonic there must correspond its image current. Mathematically this should lead to the appearance of a form factor $g(k)$ (where $k = \omega/c$) in the field of the image. Without knowing $g(k)$ in concrete form, it is still possible to confirm that $g(k)$ tends to zero when the dielectric constant ϵ tends to 1. Hence it is possible to estimate the order of the most important k and thus the order of the spreading.

Let us assume, for example, that the dielectric constant of the plasma is equal to $\epsilon = 1 - \omega_0^2/\omega^2$, then $g(k)$ tends to zero when $\omega >> \omega_0$, that is the spreading Δk is of the order of ω_0/c. From the uncertainty relationship we find the order of the spreading of the image $\Delta z \Delta k \sim 1$; $\Delta z \simeq c/\omega_0$.

If the loop is less than distance Δz from the plasma, then evidently the potential energy begins to change slightly, reaching its maximum value at the boundary. For $z >> \Delta z$ the potential curve practically coincides with the curve for an ideally conducting wall. If z_1 is the point where the loop is stopped by an ideally conducting wall, then the condition for elastic electromagnetic reflection of the ring current by the plasma may be written in the form

$$z_1 > \frac{c}{\omega_0} \qquad \omega_0^2 = \frac{4\pi n e^2}{m} \qquad [1]$$

where

n = density of plasma
e, m = electronic charge and mass

The ratio of the kinetic energy of the ring to its magnetic energy before collision

$$X = Mv_0^2c^2/L_0I_0^2$$

for remote ($z_1 >> a$) and close ($z_1 << a$) collisions will be respectively much less ($\chi << 1$) and much greater ($\chi >> 1$) than unity; a is the radius, M the mass and L_0 the initial self-inductance of the ring.

Expanding in series the elliptic integrals, representing the mutual inductance of two coaxial rings of uniform radius, for remote collisions the law of conservation of energy gives

$$z_1 = a(\pi/16\Lambda\chi)^{1/3} \qquad [2]$$

where Λ is half the linear inductance of the ring before collision. In this case the reflection condition [1] is written in the form

$$n > \frac{1}{4\pi r_k a^2}\left(\frac{16\chi\Lambda}{\pi}\right)^{2/3} \qquad [3]$$

where $r_k = e^2/mc^2$ is the classical electron radius.

When $a = 10$ cm and $\chi = \frac{1}{16}$, we get $n > 10^{10}$ cm^{-3}. For close collisions $\chi >> 1$ it is possible to use the mutual inductance of two parallel ideal conductors with currents flowing in opposite directions

$$L - M = 4\pi a \ln\left(\frac{z}{r} + \sqrt{\frac{z^2}{r^2} - 1}\right) \qquad [4]$$

Translated from *Vestnik Moskovskogo Universiteta* (Bull. of Moscow University), no. 6, 1959, pp. 142–145. Translated by Research Information Service.

[1] Numbers in parentheses indicate References at end of paper.

where r is the radius of the cross section of any of the conductors. For small distances z we can write approximately

$$L - M = 4\pi a \sqrt{\frac{2(z-r)}{r}} = 4\pi a \sqrt{\frac{2\Delta}{r}} \qquad [5]$$

Writing the criterion for reflection in the form $\Delta = z - r > c/\omega_0$, we find that reflection may occur, if

$$n > \chi/(r_0^2 r_k \Lambda^4 \pi) \qquad [6]$$

When $\chi \sim 10$ and $r_0 \sim 1$ cm, $\Lambda \sim 1$ we get $n \sim 10^{17}$ cm^{-3}.

Thus, a plasma of relatively low density can act as a good reflector of a ring current, only if the initial magnetic energy stored in the ring exceeds the initial kinetic energy.

Let us make the collision process a fast one. For this purpose we shall evaluate collision time. We shall take this time as equal to the time taken by the quantity y

$$y = (I - I_0)/I_0 \qquad [7]$$

to pass from its maximum value at the stationary point y_{max} to y_{max}/ϵ, where $\epsilon = 2,3, \ldots$. The result depends weakly on ϵ; I is the current in the loop, I_0 the current before collision.

Computations give

$$t_{CT} = \int_{y_{max}/c}^{y_{max}} \left(\frac{dz}{dy}\right) \frac{dy}{v(y)} = \frac{a}{3v_0}\left(\frac{\pi}{16\Lambda}\right)^{1/3} \chi^{-5/6} \varphi(\epsilon) \qquad [8]$$

where $\varphi(\epsilon)$ is a multiplier of the order of unity, which may be omitted from such evaluations. In obtaining Equation [8] we used the relationships

$$y = \frac{\pi a^3}{16 z^3 \Lambda} \qquad v(y) = v_0 \frac{\chi - y}{\chi} \qquad [9]$$

which hold true for remote collisions ($z >> a$); v_0 is the initial velocity of the ring.

The losses in the plasma may be counted negligibly small, if the collision time is less than the mean frequency of collisions between electrons and plasma ions. This condition may be written in the form of a limitation imposed on the conductivity of the plasma

$$\sigma_{pl} > n r_k a c^2 / v_0 \left(\frac{\pi}{16\Lambda}\right)^{1/3} \chi^{-5/6} \qquad [10]$$

When $\chi = \frac{1}{10}$, $n \sim 10^{10}$ cm^{-3}, $v_0/c \sim 10^{-2}$ we get $\sigma_{pl} \sim 10^{10}$ 1/sec.

The energy losses due to ring resistance will be substantial, if during collision the energy, liberated in the form of Joule heat, is equal to the total energy of the ring before collision, which when $\chi << 1$ may be considered purely magnetic. Hence we get the condition for the electric conductivity of the ring σ_r

$$\sigma_r > \frac{a}{3 r_0^2 \Lambda} \frac{c^2}{v_0} \left(\frac{\pi}{16\Lambda}\right)^{1/3} \chi^{-5/6} \qquad [11]$$

From Equation [7] we get an expression for the critical velocity, corresponding to the energy losses due to the ring resistance during collision attaining the amount of energy before collision

$$v_0^{cr} = c \left(\frac{ac}{3 r_0^2 \Lambda \sigma_r}\right)^{2/3} \left(\frac{\pi}{16\Lambda}\right)^{1/3} \left(\frac{L_0 I_0^2}{Mc^4}\right)^{5/16} \qquad [12]$$

The collision process will be fast when $v_0 > v_0^{cr}$.

Finally, let us evaluate the change in the major radius of the ring during collision. When $\chi << 1$ it is possible to neglect the supplementary radial action of the image. In this approximation the solution of the equation for radial motion when $\Delta a/a << 1$ gives

$$\frac{\Delta a}{a} = \frac{1}{9\Lambda} \left(\frac{\pi}{16\Lambda}\right)^{2/3} \frac{1}{\chi^{2/3}} \frac{\Lambda + 1}{\Lambda^2} \qquad [13]$$

for the change in radius during collision. The critical values of χ, for which changes in the radius during collision will be of the order of the original radius, are sufficiently small

$$\chi_{cr} = \frac{1}{84} \frac{(\Lambda + 1)^{3/2}}{\Lambda^{11/2}} \qquad [14]$$

I consider it my agreeable duty to express deep gratitude to M. L. Levin and M. S. Rabinovich for their discussion of the results.

Reference

1 Taytovich, V. N., *Zhur. Eksp. i Teor. Fiz.* (J. Exptl. Theoret. Phys.), vol. 35, no. 6, 1958, p. 1407.

—Original received May 15, 1959

Digest of Translated Russian Literature

The following abstracts have been selected by the Editor from translated Russian journals supplied by the indicated societies and organizations, whose cooperation is gratefully acknowledged. Information concerning subscriptions to these publications may be obtained from these societies and organizations. Note: Volumes and numbers given are those of the English translations, not of the original Russian.

OPTICS AND SPECTROSCOPY (Optika i Spektroskopiia). Published by Optical Society of America, Washington, D. C.

Vol. 8, no. 2, Feb. 1960.

Total Luminescence Brightness of ZnS-Cu and ZnS-Cu,Mn Electroluminophors in a Pulsating Field, by G. S. Kozina and L. P. Poskacheeva, pp. 110–111.

Abstract: Measurements of the luminescence brightness of yellow and green electroluminophors in a pulsating field are presented. The electroluminescence brightness of yellow luminophors in this mode of operation exceeds the luminescence brightness of the same layers in a bipolar field by severalfold. The brightness of green luminophors in a pulsating field is somewhat lower than their brightness in a bipolar field. The variation of brightness is connected with the occurrence of an appreciable conduction current through the layer. The increase of brightness is explained by the joint effects of electroluminescence in d-c and a-c fields.

A Photographic Method of Measuring the Optical Constants of Metals, by I. N. Shklyarevskii and A. I. Sharapov, pp. 122–123.

Abstract: A method of measuring the optical constants of metals in the visible and ultraviolet regions of the spectrum is described. Measurements of the optical constants of antimony in the visible region of the spectrum are reported.

The Determination of the Size of Particles by Scattering of Light II. Determination of the Size of Globules of the Polychloroprene Latex by Means of Optical and Electron Microscope Methods, by I. Ya. Slonim, N. M. Fodiman and Z. M. Ustinova, pp. 124–125.

Relations Between the Infrared Band Intensities of Isotopic Molecules, by L. M. Sverdlov, pp. 129–130.

The Causes of the Luminescence of Liquids Under the Action of Hard Radiation, by L. I. Al'perovich, I. D. Sherbaf and R. Marupov, pp. 132–133.

Vol. 8, no. 3, March 1960.

The Hyperfine Structure of Multi-Electron Atoms, by G. M. Bukat, A. Z. Dolginov and R. A. Zhitnikov, pp. 151–155.

Abstract: We have calculated the electronic matrix elements which enter into the hyperfine structure constants for atoms with several equivalent electrons in an unfilled shell. Cases with LS- and jj-coupling were considered. The fractional parentage coefficients for terms of maximum multiplicity of configurations f^n were given a form convenient for calculation. It was shown how, in certain cases, application of the sum rule makes it possible to solve the problem without calculating the fractional parentage coefficients.

An Approximate Method for Calculating Franck-Condon Factors, by L. M. Biberman and I. T. Yakubov, pp. 155–158.

Abstract: An approximate method is given for calculating Franck-Condon factors for diatomic molecules. The method is applicable in the region of large values of the vibrational quantum number.

Coefficient of Continuous Absorption of Radiation by Quasi-Molecules of Hydrogen, by S. P. Erkovich, pp. 162–165.

Abstract: We obtain the dependence of the absorption coefficient of quasi-molecules of hydrogen on temperature and the concentration of H atoms. It is shown that over a wide range of temperatures and pressures, quasi-molecular absorption plays an important role. A table is given from which k_λ may be evaluated under specific conditions.

The Influence of the Width of the Monochromator Slits on the Accuracy of Photoelectric Measurements of the Intensity of Raman Lines, by E. V. Chisler, pp. 187–188.

Abstract: The dependence of the mean-square error of the measurements of the line intensity on the width of the monochromator slits is examined from the viewpoint of the choice of optimum scanning conditions of the spectrum.

Determination of Absolute Oscillator Strengths From Spectral Line Widths, by V. K. Prokof'ev, I. M. Nagibina and G. P. Petrova, pp. 195–197.

Abstract: The paper describes a method for determining absolute oscillator strengths by measuring spectral line widths. The relative concentrations of Mg and Ca atoms and ions in the mantle of an arc discharge and at the electrodes were measured experimentally. The method presented was used to determine the absolute oscillator strength of the tin resonance line at 2863.33 Å; the value obtained was 0.08.

High Speed Photoelectric Spectrometer for the Region 0.4–0.9 μ, by N. G. D'yachenko, pp. 205–206.

Vol. 8, no. 4, April 1960.

The Indicatrix of the Raman Scattering of Light, by I. I. Kondilenko, P. A. Korotkov and V. L. Strizhevskii, pp. 249–252.

Abstract: A derivation is given of a simple and descriptive conclusion of Placzek's formulas, determining the dependence of the Raman line intensity on the angle of observation φ and the value of the degree of depolarization on the line ρ. The angular dependence of the intensity of the Raman lines of carbon tetrachloride, benzene and chloroform are measured. The experimental data obtained are in satisfactory agreement with theory. The possibility of a new method of measurement of the degree of depolarization of a Raman line is shown.

Infrared Spectra of Chlorophyll and Its Analogs, by A. N. Sidorov and A. N. Terenin, pp. 254–259.

Vibrational Spectra of the Modifications of Germanium Dioxide, by V. V. Obukhov-Denisov, N. N. Sobolev and V. P. Cheremisinov, pp. 267–270.

Abstract: The vibrational spectra of the vitreous and crystalline modifications of germanium dioxide were investigated. Basically, the spectra of the soluble and vitreous modifications differ little from one another; this might be a consequence of similarity of structure. At the same time, the insoluble form of GeO_2 has a completely different vibrational spectrum. This indicates the structure of the latter form to be different from the structure of the other modifications of GeO_2, which is in agreement with x-ray data.

A comparison of the vibrational spectra of the soluble modification of GeO_2 and low temperature α-quartz is made, and a compilation of the frequencies of the vibrational spectrum of the soluble modification of GeO_2 is given, arranged according to the type of vibration.

Investigation of the Absorption Spectra of Zinc Sulfide Crystals, by E. F. Gross, L. G. Suslina and K. F. Komarovskikh, pp. 273–275.

Abstract: We have investigated the polarization of the line spectrum of ZnS in the absorption region of the hexagonal monocrystalline modification at 4.2 K. We observed a slight shifting of the position of the absorption lines, associated with mechanical strain in the crystals.

The absorption spectra of films of hexagonal ZnS sublimed in vacuum were studied and compared with the spectra of monocrystalline samples of ZnS.

The Mechanism of Scintillations in Organic Substances, by I. M. Rozman, pp. 277–280.

Abstract: A brief summary is given of the theory proposed for explaining the relationship of the specific scintillation value to the ionizing capacity of a particle. A method of investigation is proposed, which considers the local density of activations and, particularly, the presence of "isolated" activations in the overall path of heavy particles, and the presence of "accumulations" of activated molecules in the path of a high speed electron.

Luminescence Properties of Cesium Iodide Crystals Grown From a Superheated Melt, by Yu. A. Tsirlin, V. I. Startsev and L. M. Soifer, pp. 283–285.

Abstract: The kinetics of the production of luminescence properties in cesium iodide crystals grown from a superheated melt is investigated. A number of original results are obtained for the explanation of this phenomenon.

The α/β Ratio in a Liquid Scintillator, by Yu. A. Nemilov, G. N. Belozerskii and A. N. Pisarevskii, pp. 292–293.

Scattering of Light in Media With Variable Optical Parameters, by A. P. Ivanov and E. B. Khodos, pp. 293–296.

Abstract: Two methods of determining the reflection and transmission coefficients of layers with optical parameters varying along the thickness are examined. For some particular cases, tables of values of reflection coefficients are compiled. Reflection spectra of mixtures of colored glass powders whose absorption constant varies with the thickness x by the law $k = a + bx$ were studied experimentally. Good agreement between experimental and theoretical data is obtained.

Use of the Phenomenon of "Total External Reflection" for Filtering the Continuous Spectrum in the Ultra-Soft X-Ray Region, by A. P. Lukirskii and Yu. A. Omel'chenko, pp. 297–299.

Abstract: A method is described for filtering out the short wave length part of ultra-soft x-radiation; this makes possible the use of the continuous spectrum as the source in the study of absorption spectra.

Vol. 8, no. 5, May 1960.

Spectroscopic Investigation of the State of the Gas Behind a Shock Wave. II, by F. S. Faizullov, N. N. Sobolev and E. M. Kudryavtsev, pp. 311–315.

Abstract: A photoelectric method was developed for measuring the temperature of the gas behind a shock wave; the method is based on the measurement of emission and absorption intensities of spectral lines. The accuracy of the method is discussed. The distribution of the shock wave velocity along the shock tube was investigated. Using the Schlieren method, data on the length of the region of hot gas and on the lag of the sodium D line emission were obtained.

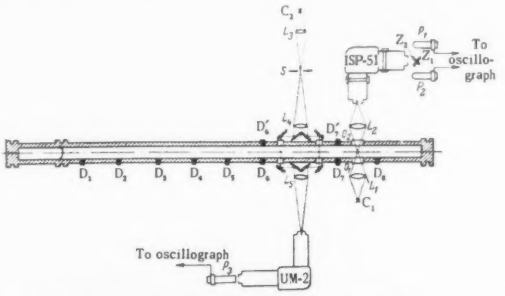

Fig. 1. Diagram of the arrangement for the spectroscopic study of the state of the gas behind a shock wave (D_1, ... D_8–ionization probes). Arrangement for investigating the shock-wave velocity with the schlieren shadow method: C_2–SVDSH-1000 mercury lamp; L_3–lens (f = 50 mm); S–slit; L_4–collimator lens (f = 800 mm); L_5–lens (f = 800 mm); P_3–FEU-19 photomultiplier. Arrangement for temperature measurements: C_1–DKSSH-1000 xenon lamp; L_1 and L_2–lenses (f = 90 mm); O_1 and O_2–windows; Z_1 and Z_2–mirrors; P_1 and P_2–FEU-17 photomultiplier.

The Temperature Dependence of the Dispersion and Absorption of Light in Molecular Crystals. I. Theory, by V. L. Strizhevskii, pp. 329–332.

Abstract: We consider the temperature dependence of the dispersion and exciton absorption of light in molecular crystals with weak exciton-phonon coupling. We obtain equations describing the temperature dependence of the refractive index and the absorption coefficient. We give several qualitative results derived from these equations.

Spectroscopic Study of the Processes of Vibrational Energy Transformation of Complex Molecules in Collisions I. On the Determination of the Amount of Energy Transferred and the Efficiency of Collisions, by B. S. Neporent and S. O. Mirumyants, pp. 336–339.

Abstract: The results of previous investigations of the authors and others on the stabilization of excited complex molecules by collisions are considered. It is shown that in studying the process of vibrational energy transformation in complex molecules in collisions with foreign gas molecules, it is necessary to take into account the exchange of energy, not only with the foreign particles but also with the translational and rotational degrees of freedom of the complex molecules themselves. A more precise expression for the accommodation coefficient is proposed and applied to data obtained in previous investigations. The results obtained are discussed briefly.

The Theory of Luminescence Quenching in Liquid Solutions, by Yu. A. Kurskii and A. S. Selivanenko, pp. 340–343.

Abstract: Equations are formulated describing luminescence quenching of solutions. Expressions are obtained for the laws of damping taking account of the Brownian motion of the molecules and of resonance quenching.

Introduction: The theoretical derivation of the law of luminescence damping for resonance quenching is generally carried out with the assumption of immobility of the radiating and quenching molecules in the solution during the lifetime of the excited state. Therefore, it is of interest to introduce into the calculations the Brownian motion of the molecules, after having first examined the particular case of "immobile" molecules. In view of the discussion on this question, it is evident that such an examination is expedient.

Luminescence of Unactivated CsI Crystals. II, by Z. L. Morgenshtern, pp. 355–357.

Abstract: The luminescence of unactivated and activated CsI crystals excited by γ-irradiation is investigated at room temperature and at the boiling point of liquid nitrogen. It is shown that three kinds of emission are observed in the luminescence spectra—ultraviolet, blue and long wave length (due to the activator)—corresponding to different luminescence centers. The spectral characteristics of the luminescence of the crystals and in particular, the temperature dependence of the luminescence are studied for all of these wave length intervals. The question of the origin of these various bands in the luminescence spectra of the crystals is considered.

A Comparison of Light Scattering Functions of Spherical Particles, by Yu. V. Mal'tsev, pp. 362–364.

Abstract: The accuracy and limits of application of different generalized parameters are examined for the graphical representa-

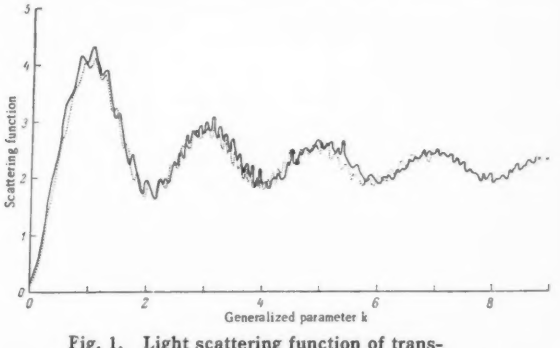

Fig. 1. Light scattering function of transparent particles with diffierent refractive indices. Dotted curve—n = 1.40; solid curve—n = 1.50.

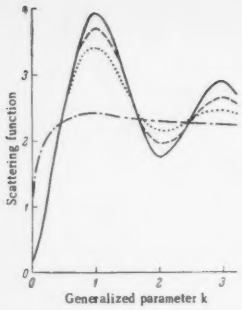

Fig. 2. Light scattering function of absorbing particles with n = 1.33 and different indices of absorption. Solid curve—x = 0; dotted curve—x = 0.03; dashed curve—x = 0.01; dashed dotted curve—x = 0.30.

tion of the scattering function in which the positions of the respective fundamental extrema of the different functions coincide. A new generalized parameter is proposed, the application of which for the calculation of the positions of the extrema in the normal construction of the functions does not require any information about its functions. The scattering functions, depending on the new parameter, have a form convenient for comparison.

Study of Aberrations of a Concave Grating Under Defocusing Conditions, by A. I. Kozlenkov, pp. 365–370.

Abstract: A theoretical study of the aberrations of a concave diffraction grating and their relations to its ultimate resolving power is undertaken. We study the geometrical form of the focus near the Rowland circle, establish the dependence of particular focus characteristics on adjustment errors, obtain expressions for the aberrations in the presence of such errors, and give simple methods of evaluating the resolving power.

Passage of Light Through Lightguides, by A. R. Daich, Yu. A. Tsirlin and L. E. Pargamanik, pp. 375–379.

Abstract: The passage of light through cylindrical lightguides with specular walls is investigated in the approximation of geometrical optics. Transmissivities for lightguides with total reflection and with reflecting coverings are calculated. Account is taken of absorption of light in the material of the lightguide. The dependence of the transmissivity on the dimensions of the lightguide and on the reflection conditions at the walls is explained. A comparison is made between various types of lightguides.

PHYSICS OF METALS AND METALLOGRAPHY (Fizika Metallov i Metallovedenia). Published by Pergamon Institute, Washington, D. C.

Vol. 7, no. 5, 1959.

Theory of Diamagnetism of the Electron Gas in Crystals, by M. V. Nitsovich, pp. 1–10.

Introduction: As Landau has shown, the diamagnetism of an electron gas is bound up with the quantization of the electron orbits in the magnetic field. Landau calculated the susceptibility of a free electron, finding the value of the energy of an individual electron in a magnetic field. Peierls has shown that to calculate the magnetic susceptibility of an electron gas it is not necessary to know exactly the value of the energy of an individual electron in a magnetic field. This author was able to calculate the diamagnetic susceptibility of strongly bound electrons. Wilson generalized Peierls' method for the general case of a zone electron, using the density matrix method. Wilson was able by this to separate the principal term of the diamagnetic susceptibility of the electron gas. However, the method he used made more difficult the systematization of all the terms of the fixed part of the diamagnetic susceptibility of the electron gas as a whole. In view of the fact that in several cases recently an anomalously large diamagnetism has been observed (e.g., in semiconductors), Adams has produced a new derivation for diamagnetic susceptibility, using in his calculations Wannier's wave function apparatus.

In the present work a new derivation of diamagnetic susceptibility formulas is proposed, generalizing, it appears to us, all the preceding ones. The method we have used is based on the direct calculation of the statistical sum by means of Feynman path operator integrals. In making the calculation it was found useful to use the theory of movement of a zone electron in the K-presentation, as described in the monograph by Landau and Lifshits. Using this it was found possible to find in a quite compact way all the terms of the statistical sum, which are proportional to H^2, and the exact value of the diamagnetic susceptibility of the electron gas.

Theory of Collective Excitation of a System of Electrons in a Solid, by G. I. Guseva and G. G. Taluts, pp. 19–27.

Abstract: By a collective coordinate and impulse method the spectrum for collective excitation of an electron system in a solid is investigated. Two different branches of the energy spectrum result, one of which corresponds to plasma, and the other to exciton vibration.

Dependence of Semiconductor Parameters on Impurity Concentration, by M. Sh. Giterman and N. P. Kontorovich, pp. 37–40.

Abstract: Within a many-electron model, the energy spectrum of an impurity semiconductor was investigated. The activation energy and effective mass of current carriers in relation to the concentration of impurities was established.

Magnetic Properties of Mn_3Ge_2 Compound, by L. G. Fakidov and Yu. N. TsIovkin, pp. 47–49.

Abstract: The temperature dependence of magnetic moment of Mn-Ge alloys containing from 40 to 95 at. per cent Ge has been investigated. It is shown that the magnetization of all the compounds in a temperature range of 113–283 K is determined by the presence of one ferromagnetic phase Mn_3Ge_2 (Θ_f = 283 K). Magnetic moment M per manganese atom calculated from paramagnetic measurements is equal to 2.5 μB (Θ_p = 300 K). At 113 K the phase transition of the first kind takes place.

Conclusions:

1 In the Mn-Ge alloys containing from 40 to 95 at. per cent Ge, there is one ferromagnetic phase Mn_3Ge_2. The ferromagnetic Curie point of this phase is Θ_f = 283 K, the paramagnetic Curie point Θ_p = 300 K. The magnetic moment per atom of manganese in the Mn_3Ge_2 compound calculated from paramagnetic measurements is M = 2.5 μB.

2 For 113 K a phase transition of the first type takes place for the Mn_3Ge_2 compounds.

3 The values of specific conductivity $\rho \sim 10^{-3}\Omega$ cm and magnetic moment M = 2.5 μB indicate that bonds of metal type prevail in the Mn_3Ge_2 compound.

Magneto-Optic Resonance in Ferromagnetics II. Near Infrared Region, by G. S. Krinchik and R. D. Nuraliyeva, pp. 54–57.

Abstract: The change in the intensity of reflected light on magnetization is measured at three different angles of incidence on nickel, cobalt and iron at wave lengths of from 0.9 to 7 or 8 μ. The magneto-optic parameter is calculated for the parts of the spectrum for which the optical constants of the metal are known.

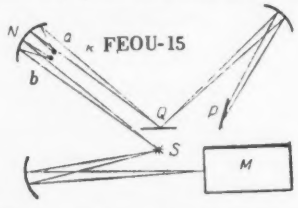

FIG. 1. Basic scheme of apparatus.

The course of the dispersion curves for the real and imaginary components of the magneto-optic parameter confirms the conclusion that there exists a new kind of resonance in the visible and near infrared region of the spectrum. The possibility of explaining this resonance as a magnetic spin resonance in an effective alternating field is discussed.

Modifications in the Structure of Nickel and of the Titanium Solid Solutions in Nickel During a High Temperature Deformation, by V. M. Rozenberg and L. V. Gradova, pp. 78–86.

Introduction: The purpose of the present work was to study the character of structural changes in nickel, and in solid solutions of titanium in nickel, during deformation at elevated temperatures (700 and 900 C). The selection of these materials for our investigations is of interest in connection with the fact that nickel forms the basis of numerous heat resistant alloys. The composition of the alloys investigated is given in a table.

In order to obtain a uniform grain size of the order of 0.2–0.3 mm, the test alloys were subjected to a preliminary annealing at the temperature of 1200 C (alloy no. 134 for 1 hr, alloy no. 136 for 2 hr and alloy no. 137 for 3 hr). The test loading was chosen in such a way that the failure of the specimens occurred several hours after the test was started or at a period of some tens of hours afterward. The deformation, at which the failure occurs within a few hours, will be referred to as the "rapid deformation," while the other, as the "slow deformation."

The specimens were subjected to tensile stresses in a vacuum testing machine. In the following, a description is given of typical changes in the structure of the alloys investigated and the main test results are reported.

Conclusions:

1 The formation of slip planes, grain fragmentation and a displacement of the grains with respect to one another were found to occur during the high temperature deformation of nickel and of the nickel-based solid solutions of titanium.

2 The grain fragmentation during slow deformation at high temperatures is not a primary breakdown process of the initial grain. It is rather associated with a prior, nonuniform deformation of the grain, which leads to the setting up of local bending stresses in the crystals. The fragmentation as found in the present test series represents a special case of polygonization, which sets in under a simultaneous action of temperature and deformation.

In the temperature interval of 700–900 C, the possibility of fragmentation is determined primarily by the speed of deformation.

3 In the process of nickel alloying with titanium, the fragmentation of grains occurs at lower deformation velocities than is the case with pure nickel.

4 The failure of specimens tested at the temperatures of 700 and 900 C has, in all cases, an intercrystalline character.

Study of Creep in Metals and Alloys. 7. Contribution to the Problem Regarding Fracture Due to Diffusion Creep, by Ya. E. Geguzin, pp. 96–99.

Abstract: A metallographic study of copper wires showing a "bamboo" structure after having undergone creep testing and having reached a stage preceding failure, has been carried out. A shift of grains along the intercrystalline boundaries and the formation of porosity along grain boundaries, has been observed. These phenomena, by bringing about a decrease of the true cross-sectional area of the specimen, lead to an increase in specific load and, hence, to the failure of the specimen.

Vol. 7, no. 6, 1959.

Effect of Current Carrier Degeneration on the Transport Phenomena in Semiconductors, by V. Ye. Vzdornov and I. M. Tsidil'kovskii, pp. 1–9.

Introduction: For many semiconductors with a high conductivity, conditions are frequently realized, under which the function of the current carrier distribution deviates from the Maxwell-Boltzmann type; in other words, the electron gas becomes partly degenerate.

The characteristics of the physical properties of degenerate semiconductors were examined in detail in another paper. Formulas and graphs for some transport phenomena in semiconductors with an arbitrary degree of current carrier degeneration for weak magnetic fields were first given elsewhere. That paper, however, contains serious errors. In deriving formulas for the adiabatic coefficients (Ettingshausen and Righi-Leduc), no account was taken of the lattice heat conduction, although in semiconductors this usually substantially exceeds the electronic heat conduction. An incorrect formula was used for the resistance variation in a magnetic field. Furthermore, all formulas are given in a form which is not convenient for comparison with experimental results, since they contain parameters (m^*, l_0) which cannot be measured directly. Rodot obtained formulas for the longitudinal Nernst-Ettingshausen effect in degenerate semiconductors for the case of electron scattering on acoustic lattice oscillations and on impurity ions. The general formula for the effect for the case of strong magnetic fields is incorrect, which can readily be seen from the particular formula for an undegenerate electron state.

In view of this, it was thought desirable to present the results of our calculations of the galvano- and thermomagnetic effects for semiconductors with any degree of degeneration of the electron gas, in both extremely weak and strong magnetic fields.

The Problem of Determination of the Diffusion Coefficient During Plastic Deformation of a Solid Body, by Iu. P. Romashkin, pp. 27–34.

Abstract: A solution is given of the problem of diffusion for a linear case in a medium undergoing deformation with the object of determining the distribution of the substance along the coordinate in the direction of which diffusion has taken place. Formulas are presented for calculating the correction factor $a = D_1/D$, where D_1 is the measured and D the required diffusion coefficient, for the case of any relation between deformation and time.

Magnetic Anomalies in Magnetite, by A. S. Mil'ner, Ye. F. Kovtyn and I. N. Popov, pp. 35–39.

Conclusions:

a On rotating the magnetic field, and on simply switching it off for a short while, in the region of 75 K the intensity of magnetization of magnetite, previously cooled in a magnetic field, decreases. This is evidently due to the fact that the anisotropy constant at 120 K and below is sharply increasing from its zero value.

b The size of the magnetization jump for magnetite below 100 K depends on the temperature at which the field is applied during the cooling process. This effect, it appears, is caused by the nonuniformly stoichiometric composition of the specimen.

Anisotropy of Crystallites and the Structure of the Debye Ring From a Stressed Specimen, by G. I. Aksenov and V. A. Moshchanskii, pp. 49–54.

Abstract: An account is given of results of an analysis of the conditions for reflection of monochromatic x-rays from atomic planes in a linearly stressed polycrystalline aggregate.

The experiment almost completely corroborates theoretical calculations and allows substantial conclusions to be drawn about x-ray measurements of residual stresses of the first kind.

Vacuum Distillation of Metals With Condensation on a Heated Surface, by V. M. Amonenko, B. M. Vasivtinskii, V. V. Lebedev and B. I. Shapoval, pp. 61–66.

Conclusions:

1 The method of vacuum distillation of iron with condensation on a heated surface enables iron with a purity of 99.98 per cent to be produced after one distillation. A second distillation reduces the impurity content of the iron to 0.01 per cent.

2 The vacuum distillation of ferromanganese enables manganese with a purity of 99.96 per cent to be produced with a pure metal yield of up to 80 per cent of the metal contained in the alloy.

Diffusion in Alloys of the System Nickel-Titanium, by A. Ya. Shinyaev, pp. 73–76.

Abstract: Examination was carried out of the diffusion of iron in cast nickel-titanium alloys in the temperature interval 950–1247 deg. The existence was revealed of minimum values of the coefficient of diffusion in the curve of its dependence on the composition of the alloy. At low temperatures this minimum is found in the vicinity of the boundary of solubility, whereas with heightened temperature, it shifts into the region of diluted solid solutions. The presence of a maximum of energy of activation of diffusion, relating to the alloys of the boundary region of solubility, is explained by the differing degree of ordering of solid solutions and the formation of structural vacancies.

Revealing the Structure of Metals by Gas Ion Bombardment, by G. V. Spivak, V. Ye. Iurassova, A. I. Klenova and T. A. Vlasova, pp. 89–94.

Abstract: In previous papers the possibility has been demonstrated for revealing the structure of metals in a broad temperature interval by means of ion bombardment; for accurately revealing the structural components of alloys by the method of ion etching; for the quantitative investigation of the depth of the boundaries between the grains of a metal by means of stereoscopic electronographs in a broad range of change of the parameters of the discharge; of finding the optimum ion bombardment regime. Certain possibilities of the IuIT-1 equipment designed in the laboratory for etching metals are discussed.

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Vol. 5, no. 1, July–Aug. 1960.

The Approximate Structure of One Class of Nonsteady Transonic Flow, by O. S. Ryzhov and G. M. Shefter, pp. 1–4.

The Cosmic Age of the Sikhote-Alin Meteorite, by É. K. Gerling and L. K. Levskii, pp. 5–6.

Introduction: The aim of the present work was to study the concentration of the radioactive isotopes H^3 and A^{39} in the Sikhote-Alin meteorite. Of course, in order to carry out a successful determination of the concentration of H^3 and A^{39}, one must have recently fallen meteorites. This is especially the case for tritium, because of its relatively short half-life. In order to measure the expected low activities, a counting system having a relatively low background was built. The natural background in the Geiger-Müller counter used for this purpose was achieved with the aid of a steel screen 15 to 25 cm thick, and a ring of counters in anticoincidence. The first of these measures excluded the soft component, and the second, the hard component of cosmic rays. In this way, the background was reduced from 360 to 10 pulses per min. A detailed description of the counting setup and the experimental methods will be given in another paper.

Synchrotron Radiation in Cometary Nebulae, by G. A. Gurzadyan, pp. 7–10.

Introduction: Cometary nebulae form a small but interesting group among the galactic nebulae. These objects are distinguished primarily by their appearance (comet- or cone-shaped), as well as by irregular variations of their brightness and structure. The bright part of cometary nebulae is usually comparable in size with the large planetary nebulae, although the star which causes it to shine belongs, as a rule, to a low-luminosity class (type A, or later); the latter circumstance is difficult to reconcile with the presence of hydrogen emission lines in the spectra of some cometary nebulae. Finally, a characteristic feature of cometary nebulae is the presence of a strong continuous spectrum, sometimes interrupted by absorption lines of hydrogen.

The peculiarities of the emission from cometary nebulae were considered seriously for the first time by Ambartsumyan, who showed that radiation emitted by these objects does not have a thermal origin. Further investigation shows that the emission from cometary nebulae can be the result of synchrotron radiation emitted by relativistic electrons moving in the magnetic field of the nebulae. It should be noted that the evaluation of this hypothesis is far from being limited to only the calculation of the relativistic electron density and the determination of the energy spectrum. The relativistic electron hypothesis is found to provide an explanation for a number of features observed in cometary nebulae, while, for example, the widely accepted hypothesis of the reflection of light from the nucleus of the nebula by dust particles does not account for these observations. Some of the results obtained by the author in connection with this problem are described in the present paper.

Ultraviolet Emission and Excitation of Oxygen Lines in the Chromosphere, by G. M. Nikol'skii, pp. 11–12.

Gravitational Instability in Flat Rotating Systems With Axial Symmetry, by V. S. Safronov, pp. 13–16.

Introduction: The problem of the present paper is to find the value of the critical density for an actual flat rotating cloud. Like earlier authors, we shall assume that the axial symmetry of the cloud is maintained at all times, and, consequently, that the disturbances are radial (annular). An examination of the condition shows that it represents a balance of the forces acting on an element which, as the result of a wave disturbance, has been displaced in the radial direction through the distance $\delta r = 1$ without change of its angular momentum relative to the center of the system. The term on the left gives the gravitation of the annular cylinder according to Poisson's equation, the first term on the right is the force acting to return the displaced element to its former orbit, and is associated with the stability of circular orbits; the second term on the right is the gradient of the gas pressure that arises as a result of the disturbance, i.e., is also a force acting to bring the element back; and the last term is very small in comparison with the others and can be neglected.

The Electron Temperature of a Medium in the Presence of Synchrotron Radiation, by G. A. Gurzadyan, pp. 17–19.

Introduction: Suppose that, in the atmosphere (or a limited volume of the atmosphere) of a star the emission of synchrotron radiation (bremsstrahlung due to relativistic electrons in a magnetic field) takes place. We assume further that the density of the synchrotron radiation in the wave length region below 912 A (L_c-radiation) is much greater than the density obtained from the Planck formula when the temperature of the star is T_*. It may then be considered that the ionization of the hydrogen atoms in the medium is entirely due to the synchrotron L_c-radiation. The electron temperature of the medium will clearly be determined by the residual energy of the electrons which are produced during the photoionization of hydrogen atoms by the synchrotron L_c-radiation, since the appearance of forbidden lines is excluded.

The present paper is concerned with the determination of the electron temperature of the medium under the following assumptions: a) Free electrons are formed by photo-ionization of hydrogen atoms due to short-wave synchrotron radiation generated in the atmospheric region under consideration; b) electrons lose all their energy through recombination processes associated with hydrogen; and, c) the electrons obey the Maxwellian velocity distribution.

The Phenomenon of Ejection by an Electrical Discharge, by V. N. Zhigulev, pp. 36–39.

Introduction: The theory of an electrical discharge in a medium of finite conductivity is examined. It is shown that an axially symmetric discharge is, in principle, accompanied by motion of the medium—the ejection effect. The laws of similarity for an axially symmetric discharge are derived, and self-similar solutions of its equations are given.

Plane Flow of a Perfectly Conducting Fluid With the Magnetic Field Nearly Parallel to the Velocity, by M. N. Kogan, pp. 40–42.

Introduction: If the magnetic field vector **H** and the velocity vector **V** in a perfectly conducting fluid are parallel at infinity, then they are parallel in the whole flow. If a fluid moving in this way flows about a body containing no magnetic field sources, then the field in the body is zero, and there is a discontinuity in the magnetic field at the surface of the body, i.e., there is a surface current. The forces acting on this current are perpendicular to the surface of the body, and can be considered as a magnetic pressure.

In plane flow, if $\mathbf{H} \neq \mathbf{V}$ at infinity, then they are not parallel anywhere in the flow, since $[\mathbf{V}, \mathbf{H}]$ = constant everywhere.[1] It is evident that, on the surface of the body

$$V_l H_n = U_\infty H_{y\infty} \qquad [1]$$

where

V_l, H_n = tangential and normal components of the velocity and the field, respectively

U_∞ = velocity of the oncoming flow which is directed along the x axis

$H_{y\infty}$ = component of the magnetic field in the direction of the y axis (see Fig. 1).

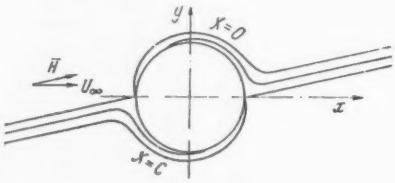

Fig. 1.

Two important conclusions follow immediately from [1].

1 $H_n \neq 0$ on the surface of the body. Consequently, there cannot be any current on the surface of a dielectric body, since, if this were not true, the forces acting on this current would be parallel to the surface, and this is impossible in a perfectly conducting fluid.

2 When the critical point ($V_l \to 0$) is approached, then the field $H_y \to \infty$.

[1] It will be assumed that an electric field, which is perpendicular to the diagram, is imposed on the flow, and that this field compensates for the induction field.

In the present work, a new type of boundary layer will be considered which is related to the first of the above indicated facts, and which arises at the surface of the body in the case when the velocity and the field are almost parallel (i.e., for $\epsilon = H_{y\infty}/H_{z\infty}$ small) in a perfect fluid that is a perfect conductor.

A Point Explosion in an Inhomogeneous Atmosphere, by A. S. Kompaneets, pp. 46–48.

Stable and Unstable Trajectories in Proportional Navigation, by V. L. Kan and A. S. Kel'zon, pp. 74–77.

Introduction: The cybernetic method of studying proportional navigation has been limited, up to the present time, to the case when the navigation constant is $b = 2$. It has been assumed, as established, that it was impossible to obtain a closed solution of even the simplest kinematic problems for $b > 2$. Locke has stated that a solution of the equations of motion can be obtained in closed form only for $b = 2$; if $b \neq 2$, numerical integration methods must be used. In engineering practice, therefore, analog computers have been applied in the solution of the problem. It is not possible, with this method, however, to obtain any overall knowledge of the problem as a whole, or to arrive at any general conclusions that are unrelated to concrete values of the parameters and definite initial conditions of the motion. Moreover, the numerical solution of the differential equations of disturbed motion does not make it possible to draw any conclusions concerning the stability of the motion. Proportional navigation is a "complex" interdependent system which does not permit investigation by the method of varying one factor at a time. It is just because of this that the more difficult method of analytic investigation cannot be discarded. We will study the problem of proportional navigation for arbitrary integral values of the navigational constant.

The Equivalence of the Rouse-Hurwitz and the Markov Stability Criteria, by Wladislaw Jarominek, pp. 78–82.

Introduction: In the present work, we started with the work of Chebyshev and Markov on continued fractions, and the relation between this work and the Rouse-Hurwitz problem that was proved by Gantmakher, and posed the problem of finding a direct relation between the Hurwitz determinants and the Markov determinants. The work led, as a result, to the formulation of a stability criterion equivalent to the Rouse-Hurwitz criterion, which we called the Markov criterion, and also, to the solution of the inverse problem of the stability of a linear system. For this, the latter was treated as the set of values of the coefficients of the characteristic equation which satisfy the conditions for the desired degree of stability relative to the determinants.

The Question of Convectional Instability of a Plasma, by Yu. A. Tserkovnikov, pp. 87–90.

Introduction: In (1)[2] the convectional instability of a plasma, i.e., instability which arises due to spatial inhomogeneities of the parameters by which it is characterized, has been discussed in terms of the kinetic equations, neglecting collisions between particles. Such an approach is reasonable in the investigation of a low density plasma, although the resultant equations are highly complicated. In the present paper, we consider the same problem as in (1), proceeding, however, from the simpler hydrodynamic equations. In solving the equations, we shall restrict ourselves to the study of the behavior of small scale perturbations, neglecting the boundary conditions, a justifiable approach in the study of convectional instability.

Radiation Measurements During the Flight of the Second Moon Rocket, by S. N. Vernov, A. E. Chudakov, P. V. Valukov, Yu. I. Logachev and A. G. Nikolaev, pp. 95–99.

General Hydrodynamic Theory of a Supersonic Interferometer, by I. I. Ol'khovskii, pp. 105–109.

Introduction: In one previous paper we proposed a method of solution of the general equations of hydrodynamics, and in another we considered the question of boundary conditions of the Maxwell type. On the basis of this work, in a third paper we solved the problem of a supersonic interferometer for the four-moment approximation, i.e., without taking heat conduction into account. In the present work, we will give a general hydrodynamic theory for a supersonic interferometer for a 5-moment approximation, i.e., for an approximation that takes the heat conduction of the gas and the boundary into account.

[2] Yu. A. Tserkovnikov, *Zhur. Tekh. Fiz.*, vol. 32, p. 67, 1957. (*Soviet Physics–Technical Physics*, vol. 5, p. 58.)

Variational Principles in the Determination of the Fundamental Characteristics of Ferromagnetic From an Analysis of the Hysteresis Loop, by V. I. Skobelkin, pp. 122–124.

Approximate Solution of the Problem of Motion of a Conducting Plasma, by G. A. Skuridin and K. P. Stanyukovich, pp. 132–136.

Radiation Pyrometry of Metals in the Near-Infrared Region of the Spectrum, by D. Ya. Svet, pp. 141–143.

Stress Relaxation and Creep as Processes of Viscous Flow, by N. S. Fastov, pp. 144–147.

The Inhomogeneity of Plastic Deformation in Extension, by B. M. Strunin, pp. 151–154.

The Thermodynamics of Irreversible Processes in Plastic Deformation, by N. S. Fastov, pp. 160–163.

Introduction: In the theory of relaxation processes, it is tacitly assumed that the relaxation-time spectrum of the body remains constant during the deformation, and this is equivalent to neglecting the variation of the macrostructure of the body and, in particular, the plastic deformation. In plastic deformation, the relaxation spectrum corresponding to an increasing load differs from that corresponding to a decreasing load, as is shown by the change in macrostructure during the deformation.

We will consider a small elastic-plastic deformation. In the deformation process, both the relative volume ϵ_{ee}, and the pure shear deformation $\epsilon_{ik} - \frac{1}{3}\epsilon_{ee}\delta_{ik}$ usually vary. However, in view of the fact that the volume of the body does not vary during the plastic deformation occurring after the removal of the load, we will limit our investigation of the general properties of plastic deformation to the case of pure shear.

Effect of X-Ray Irradiation on Tissue Respiration, by E. N. Kuznets, pp. 183–184.

Introduction: Our aim in the present work was to investigate the changes occurring in tissue respiration following the exposure of the organism of a lethal dose of x-rays. As objects of study we took: The spleen, as one of the most radiosensitive

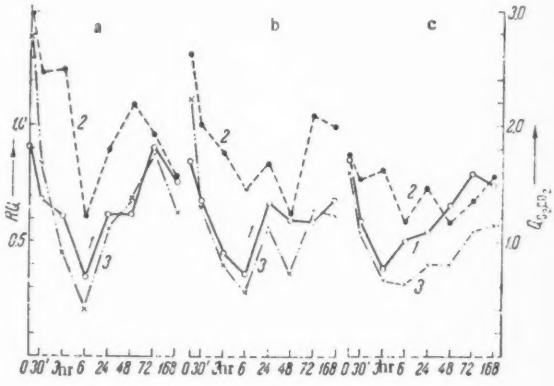

Fig. 1. Oxygen uptake (Qo_2), carbon dioxide evolution (Qco_2) and respiratory quotient (RQ) of tissues of white rats before and after exposure to X-rays (800r) (from mean data). a) Spleen, b) liver, c) skeletal muscle. 1 - RQ; 2 - Qo_2 in milliliters per 1 mg dry weight per 1 hr; 3 - Qco_2 (in same units).

tissues; skeletal muscle, as one of the least radiosensitive tissues, and liver, as a tissue of mean degree of radiosensitivity. We also took into account the fact that the gas exchange of these tissues is an important element in the total gas exchange of the organism.

Increase of Resistance of Animals to Action of Ionizing Radiation as a Result of Previous X-Ray Irradiation, by N. I. Nuzhdin, M. D. Pomerantseva and N. N. Kuznetsova, pp. 203–205.

Introduction: The aim of the present study was to investigate the nature of the change in the radiosensitivity of animals

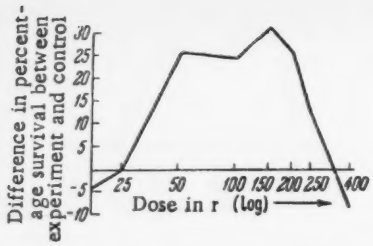

Fig. 1. Effect of preliminary doses of irradiation on the survival of irradiated mice (600 r).

resulting from their preliminary irradiation, and also to find out the relationship between the degree of protection and the experimental conditions. The work was carried out on white mice, mainly males 2.5 months old. In the first set of experiments we investigated the dependence of the protective effect on the size of the preliminary dose and the interval between the first and second exposure. For the preliminary irradiation we used eight doses: 15, 25, 50, 100, 150, 200, 250 and 400 *r*. In the second exposure the animals were irradiated with a dose of 600 *r* at different intervals after the preliminary irradiation.

RADIO ENGINEERING (*Radiotekhnika*). Published by Pergamon Institute in conjunction with Massachusetts Institute of Technology, Cambridge, Mass.

Vol. 15, no. 1, 1960.

Design of Chebyshev-Type Stepped Junctions, by A. L. Fel' dstein and L. R. Yavich, pp. 1–18.

Introduction: A large number of articles are devoted to the construction of stepped junctions between two transmission lines with different wave impedances. The development of the theory in this field has advanced from the approximate representations to more precise ones. At present the construction of stepped junctions raises two main problems: Systematization of the engineering calculation of stepped junctions; tabulation of typical results in order to reduce the time spent on the calculation to a minimum.

In the article both of these problems are examined. A systematic exposition of Kollen's method is given. As the main result of the work, a catalog of 405 typical problems solved by this method is given. The catalog ensures the solution of nearly all practical variants of stepped-junction calculations.

Effect of Frequency and Amplitude Modulated Oscillations on Linear Systems, by I. T. Turbovich, pp. 39–46.

Abstract: A method is suggested for the calculation of the effect of frequency and amplitude modulated oscillations on linear systems. It is shown that, as opposed to the static, the dynamic transfer factor of the system depends not only upon the frequency of the input voltage but also upon the derivatives, with respect to time, of the frequency and the amplitude.

A Method of Calculating the Correlation Function at the Output Terminals of a Nonlinear System, by I. M. Tepliakov, pp. 49–57.

Abstract: It is shown that in order to calculate the correlation function at the output of a nonlinear system with normal noise acting on the input, it is sufficient to compute the spectrum at the output of that system with two sinusoidal oscillations acting.

The Comparative Noise Immunity of a Two-Channel Correlated Receiver and a Receiver With a Square-Law Detector, by V. S. Voiutskii, pp. 95–100.

Abstract: The comparative noise immunity in actual reception conditions for a two-channel correlated receiver and a single-channel receiver with a squarelaw detector is evaluated.

Vol. 15, no. 2, 1960.

The Principles of the Construction of Reading Machines, by A. A. Kharkevich, pp. 1–14.

Abstract: In this article certain general problems pertaining to the construction of reading machines, i.e., a machine which can automatically recognize letters and figures from their contours, are posed and briefly discussed.

The Generalized Operator Characteristics of Filters and Modulators, by M. Ia. Kaller, pp. 53–66.

Abstract: Network theory methods are extended to more complex four-terminal networks, such as filters, delay lines and modulators. Objects of the theory of linear operators are examined which serve as the mathematical expression of the concepts of filters and modulators.

Conclusion: The introduction of the generalized operator characteristics of filters and modulators shows that to each unit expansion, i.e., to the possible expansion of a signal into components, corresponds a filter-modulator pair. The idealized characteristics of these devices are derived from the generalized characteristics by the substitution of the appropriate unit expansion.

When such an approach to the characteristics of filters and modulators is adopted, we establish with maximum detail the frequency-time symmetry, and facilitate the transfer of methods for the construction of systems with frequency multiplexing into systems with time multiplexing, and vice versa; the possibility is also revealed of utilizing these methods for expansions other than those of frequency and time. The operator characteristics may assist in a more correct estimation of the functions performed by a unit of a communications system, by forming the equivalent circuit of filters, modulators and amplifiers in a similar manner to that by which the equivalent circuit of a choke comprising *r*, *L* and *C* elements facilitates the forecasting of the nature of the processes which take place in the circuit.

Certain Relationships in Optimum Systems of Signal Detection, by L. S. Gutkin, pp. 73–88

Introduction: Approximate relationships for determining the sensitivity of optimum systems of signal detection against a background of white noise are given. The error in the relationships obtained does not exceed 0.5–1 db (provided detection error probabilities do not exceed 10 per cent) and tends asymptotically to zero when these probabilities are reduced.

The analysis is carried out for binary (or single channel) and multialternative (or multichannel) detection of three kinds of signal: A signal which is known exactly, a signal with random phase and a slowly fluctuating signal. The different cases of detection are compared.

Some Problems of the Theory of Reliability of Electronic Equipment, by B. R. Levin, pp. 103–116.

Abstract: The general relationships for determining the magnitude of the reliability of a system are derived. Asymptotic estimations of the quantities which characterize the reliability of systems for exponential, Rayleigh and normal distribution laws for the reliability of the components are given. A method is described for taking into account the time involved in replacing the rejected components.

RADIO ENGINEERING AND ELECTRONICS (*Radiotekhnika i Elektronika*). Published by Pergamon Institute in conjunction with Massachusetts Institute of Technology, Cambridge, Mass.

Vol. 4, no. 1, 1959.

Some Questions on the Dispersion of Radio Waves in the Ionosphere, by V. D. Gusev, pp. 15–22.

Abstract: The part played by nonuniform waves in the angle spectrum of a dispersed field is ascertained when the ionosphere is illuminated by plane and divergent waves. The full time-space correlation function is given.

Conclusion: As a result of the absence of frequency response (dependent on ω) of the dispersed field parameters in the ionosphere, in vertical sounding one must assume that wave dispersion takes place in a limited area of its reflection. But with statistical analysis of the dispersed field it is essential to interpret the results of III, in which the complete form of the correlation function of a field (II) was given, taking into account the ex-

tremities of distance of the radiating and receiving antennas from the ionosphere and different movements and variations taking place in it.

As from experimental data the radii of correlation of the dispersed field satisfy the inequality $a_z > \lambda$, then in all cases one can really disregard the influence of nonuniform waves.

It is essential to note that, if when the ionosphere is illuminated by a divergent wave the dispersed field is statistically nonuniform in space, then in time it is statistically stationary at all points in space (naturally under conditions of statistical immobility of the dispersing layers of the ionosphere). This is also satisfied by the law of substitution of intermediates central in time under the conditions of experimental research on the fine structure of the ionosphere.

The Investigation of Winds and Nonuniformity in the Ionosphere by a Radio Astronomical Method, by V. V. Vitkevich and Iu. L. Kokurin, pp. 23–29.

Abstract: A radio astronomical method for the study of nonuniform structure in the ionosphere and motion in it is described. The results of preliminary measurements of velocities and directions of ionospheric winds are described, as worked out in a simplified manner.

Apparatus:

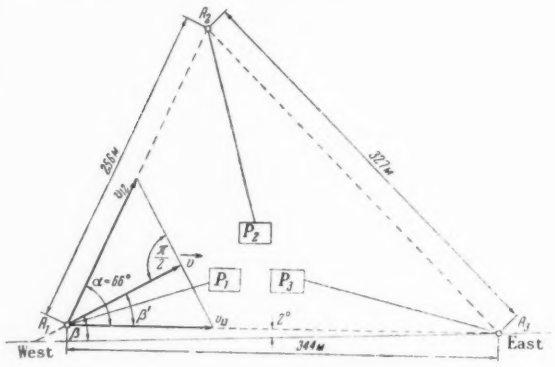

FIG. 1. Diagram showing set up of antennae, A_1, A_2, A_3, = antennae, P_1, P_2, P_3 = receivers.

Aerials—During the investigation three identical stationary aerials were used on the wave length $\lambda = 6$ m set up in a meridian plane at an angle $\delta = 49.5$ deg. Each aerial took the form of a paraboloidal grid reflector with a square aperature having an area of 170 m². The width of the directivity curve of the aerial (at half power) is 21 deg. The focal length of the paraboloid is 6.7 m. At the focal point of the reflector there is a half-wave dipole and at a distance of 0.2 λ from it a half-wave parasitic reflector. A diagram showing the positioning of the aerial is given in Fig. 1.

Receivers—All three receivers were placed at one spot (Fig. 1). This simplifies operation of the installation and made it possible to record all three on one tape.

Each receiver contained a push-pull high frequency amplifier, oscillator, mixer, four-stage intermediate frequency amplifier, detector, balanced push-pull direct current amplifier and recording equipment. The intermediate frequency of the receiver was 10 mc per sec bandwidth 0.4 mc per sec and with an output time constant of half a second. The anode and heater supply lines of each receiver were stabilized (stabilization factor: $\sim 10^{-3}$ for the anode and $\sim 5.10^{-2}$ for the heaters).

Absorption of Centimeter Waves in the Atmosphere, by S. A. Zhevankin and V. S. Troitskii, pp. 30–41.

Introduction: Constant (not sporadic) absorption of centimeter waves in the atmosphere is caused by water vapor and oxygen. The calculation of absorption where propagation takes place between two points at different altitudes presents certain difficulties owing to the dependence of the coefficient of absorption on the altitude and owing to the fact that the sphericity of the atmosphere and the refraction taking place in it, have to be taken into account. As far as is known, up to the present time absorption calculations have been made only for the case of a plane atmosphere, neglecting refraction. It follows that the calculation is of little value for general use and, in addition, an ideal plane atmosphere leads to substantial errors which arise from the angle of elevation of the point above the horizon being diminished and the distance increased.

In the present article complete calculations are given for absorption in the atmosphere, the curvature of Earth and refraction being taken into account. The general expressions derived enable the absorption of centimeter waves to be determined for a given temperature and absolute humidity at the surface of Earth.

Signal Detection Against a Background of Normal Noise and Random Reflections, by V. D. Zubakov, pp. 42–59.

Abstract: This article is devoted to the theory of optimum detection of radar signals in the presence of noise and reflections from local objects distributed at random. Stray reflections are represented as a random process for a train of main pulses. The intra- and inter-period treatment of the input signals are examined.

The Spectrum of Flicker Noise, by A. N. Malakhov, pp. 86–100.

Abstract: A short survey is given of the basic experimental and theoretical results concerning flicker noise in various systems and the fundamental difficulties in the theory of flicker noise are discussed.

The possibility of overcoming these difficulties by assuming a nonstationary nature for flicker noise are examined. Considerations are put forward concerning the causes and localization of flicker noise.

An Analysis of the Operation of Series Ionic Converters, by A. E. Slukhotskii, A. S. Vasil'yev and V. M. Martsinovich, pp. 101–113.

Abstract: An analysis is given of the operation of a series ionic converter which can be used in a number of cases as a source of high frequency current. The conditions for obtaining a voltage curve near to sinusoidal across a resistive load are determined, the regime of operation being stable. The latter is a point in favor of the circuit described compared with the more widely used parallel ionic converter.

Growing Waves Propagating in a Plasma, by G. F. Filimonov, pp. 122–140.

Abstract: A unique solution of the one-dimensional linearized kinetic Vlasov equation describing the propagation of high frequency signals generated by an external signal is found. Electron beams with a square Maxwellian electron velocity distribution in the undisturbed state are examined as examples. It is shown that for small electron-gas temperatures the usual single velocity approximation can be used. Analysis of the expressions obtained made it possible to establish the propagation direction of the natural waves in the plasma and to solve the problem of growing waves in a rectilinear electron flow.

An Electrostatic Lens Analyzer of Electron Velocities With a High Resolving Power, by A. N. Kabanov and V. I. Miliutin, pp. 176–191.

Abstract: The properties of a single cylindrical electrostatic lens operating as a velocity analyzer are described. The results of an experimental investigation into the dependence of the dispersion and resolving power of the analyzer on the geometrical parameters of the lens, the distance of the slot from the lens axis, the electron velocities, etc., are given. The construction and the supply circuit of an analyzer with a resolving power of 60,000:1 are described.

Localizing Irregularities in the Ionosphere, by A. A. Gorozhankina, pp. 212–213.

Potential Accuracy in the Reading of Angular Coordinates in Radar Systems, by S. E. Fal'kovich, pp. 237–242.

Calculation of potential dispersion of values: It is proposed that reliable detection be effected of a reflected signal against a background of fluctuating noise. This task consists of the determination of the limiting accuracy as regards the reading of the angular coordinate θ_0 use being made of the single-pulse method. The dispersion $\overline{(\theta_0^* - \theta_0)^2}^*$ of the value θ_0^*, in relation to the true magnitude of the coordinate θ_0^*, serves as a measure of accuracy.

The most that can be done with a receiving apparatus, as regards the measuring of an angular coordinate, is to establish, in accordance with $X(t)$, the oscillation received and in the capacity of output effect, the distribution of a posteriori probabilities $Px(\vartheta)$ concerning the presence of a target at all angular coordinates ϑ within a certain area $(\vartheta_1;\ \vartheta_2)$. The a priori distribution of angular coordinates within this area is suggested as being proportionate, and the coordinate of distance as being fixed.

The Nuclear Quadrupole Radio Spectrometer, by E. I. Fedin and G. K. Semin, pp. 204–206.

Introduction: Our aim has been to discover a scheme for observing quadrupole resonance which would make maximum use of standard equipment.

After protracted experiments it was found that satisfactory results could be obtained on a quadrupole radio spectrometer, the block diagram of which is shown in Fig. 1.

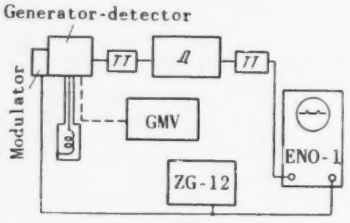

FIG. 1

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The Application of Sequential Analysis to Binary Communication Systems With a Rayleigh Distribution of Signal Intensity Fluctuations, by A. E. Basharinov and B. S. Fleishman, pp. 1–9.

Abstract: Systems for processing fluctuating signals, using sequential analysis, are considered.

The average time characteristic and the probability of correct signal detection are calculated for the case of transmission with passive spaces, when the probability of false response (false alarms) is small in comparison with the permissible probability of missing the signal.

The probability distribution of the number of completed tests is evaluated.

The Synthesis of Atmospheric Forms and the Effective Parameters of the Lower Part of the Ionosphere at Low Frequencies, by Ia. L. Al'pert and D. S. Fligel, pp. 67–77.

Abstract: The results of theoretical calculations of the form $E(t, r)$ of atmospherics and their comparison with signals received at distances of $r \sim 500$–3000 km from the source are briefly described. Theoretical values of effective conductivity previously obtained using an ionospheric model in the form of transition layers, characterized by the relationship of electron density N to the number of collisions ν dependent on height z, were used. The close resemblance of temporal variation of theoretically synthesized signals $E(t, r)$ to the observed ones is noted.

Space-Charge Wave Amplification in an Electron Beam Moving With Variable Velocity in Channels in a Lossy Medium, by Iu. F. Filippov, pp. 99–106.

Abstract: Phenomena occurring in the propagation of a modulated beam with variable velocity in channels in a lossy medium are studied theoretically. For the special case of the dependence of velocity on coordinate $v_0(x) = u_0 e^{-ax}$ expressions are obtained for exponentially increasing waves.

The Three-Cavity Klystron Multiplier, by A. D. Sushkov, pp. 124–132.

Introduction: A three-cavity klystron multiplier with signal preamplification is a frequency multiplier in which the first and second cavities are designed for the input (fundamental) frequency, and the third cavity for the output (multiplied) frequency. The signal applied to the input of such a klystrom is first amplified in intensity, and then the frequency is multiplied.

From the literature only a few works are known which are devoted to three-cavity klystron multipliers. Descriptions of the design and certain parameters of operating multipliers are given in them.

For the purpose of explaining the particular features of three-cavity klystron multipliers and their advantages over two-cavity multipliers, theoretical and experimental investigations were carried out on these devices. Certain results of these investigations are given in the present article.

Frequency Conversion in Reflex Klystrons, by E. N. Bazarov and M. E. Zhabotinskii, pp. 133–145.

Abstract: The possibility of realizing frequency division and multiplication in reflex klystrons is shown theoretically.

Studies are made of operation in resonance of the second kind and second-harmonic synchronization, fractional resonance, subharmonic synchronization and frequency multiplication by extraction of harmonic components from electron bunches formed at the fundamental frequency.

All these operating conditions have been studied experimentally. Satisfactory agreement with theory was obtained.

To increase the coefficient of electron interaction in frequency conversion a new design of reflex klystron is proposed with two cavity resonators, having three or four grids. In the latter case there is a drift space between the cavities.

The Possibility of Further Generalization of Single-Circuit *LC*-Oscillators in Quasi-Linear Theory, by P. G. Korolev, pp. 146–158.

Abstract: A single-circuit LC oscillator with an external harmonic emf introduced into the control grid circuit of the tube is considered.

A general expression is obtained for the complex amplitude of the first harmonic of the oscillator anode current, valid for arbitrary form of tube characteristics and for arbitrary values of external emf amplitude, synchronous or asynchronous. It is shown that the state of synchronous oscillation is the most general form of operation of the LC oscillator including in itself both self-oscillatory and asynchronous effects. The abridged differential equations of the synchronized oscillator are investigated.

The Effect of an emf With Periodically Varying Parameters on an Oscillating System, by N. N. Lunacharskii, pp. 179–189.

Abstract: The work considers phase variation under the action of an emf with periodically varying amplitude and frequency on an oscillating system (weak signal case). The considerations are carried out on the basis of an analysis of the Ricatti equation with variable coefficients. The work contains an analysis of the locking conditions and nonstationary processes in the general case as well as a more detailed analysis of the locking conditions in the case where the emf constitutes a sequence of coherent pulses.

Longitudinal Electron Motion and Tolerances in Linear Accelerators, by A. D. Vlasov, pp. 190–200.

Abstract: Certain questions are considered of particle dynamics in high energy linear electron accelerators. The conditions of injection, insuring effective acceleration and compact energy spectrum of the accelerated particles, are formulated. An estimate is made of the influence of phase velocity errors on the electron output energy, permitting the establishment of tolerances for frequency, temperature and dimensions of a wave guide accelerator.

Radiation Emitted During the Passage of Charged Particles Near Ideally Conducting Bodies, by Iu. N. Dnestrovskii and D. P. Kostomarov, pp. 201–214.

Abstract: The general problem of the radiation of electromagnetic waves during the passage of charged particles in vacuum near ideally conducting surfaces is considered in the nonrelativistic approximation. Expressions are obtained for the energy and radiation spectrum. The special problem of the excitation of oscillations on entry of electrons for an open half-space into round wave guide is studied in detail.

Limiting Sensitivity and Noise Spectrum of Germanium Junction Photodiodes, by L. Ia. Pervova, pp. 237–243.

Introduction: The results of investigations on the limiting sensitivity, absolute magnitude and spectral distribution of noise of germanium junction photodiodes developed in the Leningrad Physico-Technical Institute are presented.

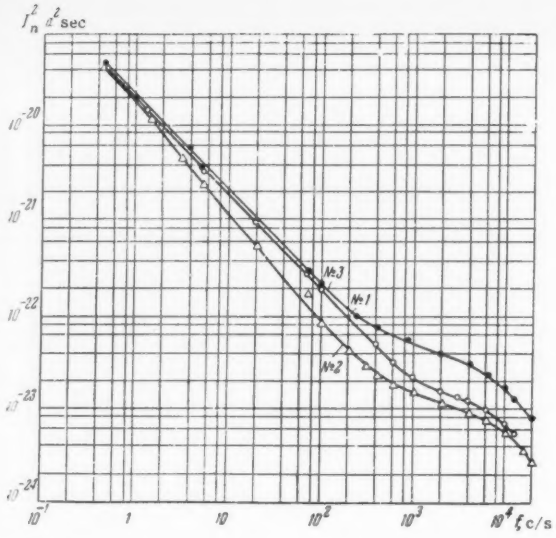

Fig. 2 Noise spectra of germanium photodiodes

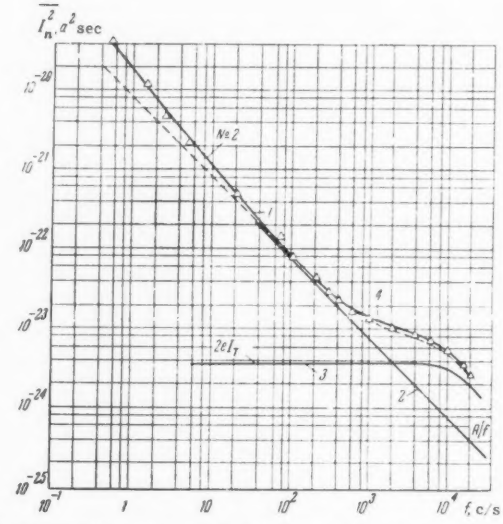

Fig. 3 Representation of spectral distribution curve in the form of two components in accordance with formula (1):

(1) experimental curve; (2) curve of the form $1/f$; (3) shot noise curve; (4) sum of curves 2 and 3.

The threshold of sensitivity of the photodiodes was of the order of 10^{-10} lm. $sec^{1/2}$ for a modulation frequency of the light flux of 70 cps. The noise spectrum of the photodiodes in the band from 0.5 cps to several hundred cps was subject to a $1/f$ distribution. For higher frequencies the noise was close to the value of shot noise of the photodiode dark current.

The Application of Wave Guide Matrices for the Calculation of Cross-Symmetrical Four-Poles, by L. R. Iavich, pp. 255–259.

Introduction: A method of calculating four-poles using wave matrices, considered previously, may be substantially simplified in the presence of vertical symmetry in the matrix.

Bombardment Induced Conductivity of Antimony Sulphide Films, by Ia. A. Oksman and G. P. Tikhomirov, pp. 260–264.

Introduction: Recently attempts have been made to utilize the increase in conductance of thin dielectric and semiconductor films under the action of electron bombardment, called here cathode conductance, for amplification of photo-emissive currents. Such attempts are justified, since currents induced in cathode conductance substantially exceed currents realized in a single stage of secondary emission amplification. The ratio of excited current to excitation current, which will further be termed amplification, in the case of cathode conductance may be exceedingly large. Thus in layers of arsenic sulphide at 163 C it is equal to 40,000; at room temperatures the amplification does not ordinarily exceed 100.

The purpose of the present work was to define the limiting values of amplification which may be obtained from these films at room temperature. In addition it was desirable to estimate relaxation time of cathode conductance and draw conclusions on the possibility of using antimony sulphide films in "ebicon"-type devices.

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Signals With Maximum and Minimum Probability of Detection, by V. A. Kotel'nikov, pp. 5–11.

Abstract: This article gives those signals which have maximum and minimum probability of detection by an ideal receiver. The evaluation of these probabilities is given.

Some Problems of the Spectral Correlation Theory of Non-Stationary Signals, by N. A. Zheleznov, pp. 12–30.

Abstract: For the description of nonstationary signals we introduce energetic characteristics connected with the present moment of time, and time-average characteristics; also frequency correlation functions characterizing the correlative relationship between the harmonic components of a random spectrum. Relationships are formulated between the corresponding spectral and correlation functions in the form of Fourier transformation pairs. The ergodic theorems are proved for nonstationary signals, and the physical content of the conditions for which these theorems are valid are explained. The spectral and correlational functions of a few classes of nonstationary signals are examined.

Influence of Noise Fluctuations on a Decoding Unit During the Intervals Separating the Pulses in a Code Group, by L. P. Kuklev, pp. 31–40.

Introduction: A large number of articles is devoted to the effect of interference on the reception of coded signals. Thus, for example, in references cited the effect of irregular pulse noise on the appearance of spurious responses of the decoding unit and on the signal suppression are examined.

In the present article we investigate the problem of the effect of noise fluctuations in radionavigation systems with pulse interval coding. Pulse interval coding comprises the transmission of n pulses identical in form and duration with a certain definite position of the pulses relative to each other. Decoding of such signals is often carried out with the help of delay lines and coincidence stages, by comparing the code intervals in the signal with the delay intervals in the lines which are adjusted for the reception of codes fixed in advance.

The investigation is carried out for the case when, at the output of the IF amplifier, there is an amplitude detector operating under the conditions of linear, inertia-free detection, and an amplitude limiter which senses a pulse signal if the voltage, at the moment when the signal appears, exceeds the limiting level.

Absorption of Radio Waves in Resonance Regions of an Inhomogeneous Plasma, by N. G. Denisov, pp. 51–65.

Abstract: With reference to the problem of the propagation of electromagnetic waves in an inhomogeneous magneto-active plasma, we investigate the passage of waves through a region where the refractive index becomes infinite. In this region the frequency of the external field coincides with the natural frequency of plasma oscillations. A calculation of the effect of absorption of the extraordinary wave in the resonance region is carried out neglecting the effect of interaction. The absorption of the ordinary and extraordinary waves is also investigated taking into account their interaction. The results obtained are discussed with reference to the case of wave propagation in the ionosphere.

The Use of Turbulence Theory to Investigate the Scattering of Radio Waves From Wandering Inhomogeneities, by M. I. Rodak and A. V. Frantsesson, pp. 66–74.

Abstract: Scattering of radio waves from particles, the motion of which can be described in terms of Kolmogorov and Obukhov's turbulence theory, is investigated.

The conditions are clarified, which make it permissible to

evaluate the correlation function (the spectrum) of the scattered field (of the components of its amplitude) and the correlation function (the spectrum) of its intensity, while assuming the velocity of the scattering centers to be constant.

The field spectrum (the spectrum of the components of its amplitude) and the intensity spectrum are compared.

Qualitative Analysis of an Nonlinear Differential Equation of the Second Order of a Self-Oscillating System With Bounded Increment Region, by A. A. Luchina, pp. 127–139.

Abstract: The nature of the solutions and the type of the limit cycle of a self-oscillation equation are investigated for the case of large nonlinearity, by constructing the phase portrait of the system. The small parameter method is used.

Conclusions: The article gives a mathematical justification for the picture of the oscillations of a system with bounded increment region given on the basis of qualitative arguments by Teodorchik, whereby the behavior of a self-oscillating system, as determined by the position and form of the limit cycle, is related to the division of the phase plane into increment and decrement regions and the form of the isocline of the horizontal tangents.

The nonlinear differential equation

$$\ddot{x} - 2\epsilon\left(1 - \frac{x^2}{a^2} - \frac{\dot{x}^2}{b^2} - 2\frac{x\dot{x}}{d}\right)\dot{x} + x = 0$$

for large values of the parameter ϵ, by constructing the system phase portrait, defined by the equation

$$\mu\frac{dy}{dx} = \frac{2\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2} - 2\frac{xy}{d}\right)y - \mu x}{y}$$

where $\mu = 1/\epsilon$.

Noninertial Envelope Transformation of Quasi-Harmonic Fluctuations, by I. N. Amiantov, pp. 140–149.

Abstract: The two-dimensional moment $[G_1(A_1)G_2(A_2)]$ is calculated for noninertial transformations of G_1 and G_2 of the envelope of quasi-harmonic fluctuations $A_1 = A(t)$, $A_2 = A(t + \tau)$. Cases of stepwise linear and exponential transformations of G_1 and G_2 are considered. In two particular examples the results are illustrated by graphs.

The Nonlinear Theory of the Two-Beam Electron Tube. Part 1. Deduction and Investigation of Equations, by G. F. Filimonov, pp. 196–209.

Abstract: The nonlinear equations for a multibeam tube and the laws of conservation derived from them are deduced and studied. These equations are reduced to the same form in linear approximation as the Vlasov kinetic equation. The dependence of the active radius of repulsive forces on the parameters of the tube is studied.

Experimental Investigation of Charge Formation on the Surface of a Dielectric Under the Influence of Electron Bombardment. Part II, by N. M. Karnaukhova and V. Ia. Upatov, pp. 237–245.

Abstract: The impulse method for measuring potential distribution is used to study the kinetics of dielectrics being charged by an electron beam. Reliefs showing the potential distribution of positive and negative spots are given. It is shown that the kinetics of the formation of a small positive spot charge differs from the kinetics of the formation of a small negative spot charge.

Conclusion: The results of the present work can be summed up as follows:

The kinetics of charging up dielectric material by an electron beam was investigated by the impulse method for measuring the potential and the charging current.

Experimental curves were obtained for the potential distribution on the dielectric surface exposed to the bombardment by an electron beam (potential relief maps for positive and negative spot charges).

It is shown that the kinetics of the formation of a small positive spot charge differs from the kinetics of the formation of a negative spot charge. This difference comes from the fact that the potential of a small positive spot charge tends to a quasi-equilibrium value, which is smaller than that of the collector potential, whereas the potential of a small negative spot charge tends to a value similar to that of the collector potential.

In previous work, based on the study of a small insulated metal electrode potential, the author has come to similar conclusions on the potential of a small positive spot.

The value of the potential in the center of the spot, which corresponds to the inflection point on the curve of the charging current at the formation of a small positive spot, is considerably smaller than the collector potential.

The difference between the kinetics of the formation of a positive and a negative spot charge becomes smaller when a fine mesh grid situated near the surface of the dielectric is used as collector. The values of the potential at the center of both a positive and a negative spot approach the corresponding values of the collector potential.

Density of Distribution of Intervals Between Zeros of a Narrow Band Normal Stationary Random Process, by L. I. Bialyi, pp. 266–269.

Probability Density of the Phase Derivative of the Sum of a Sinusoidal Signal and Gaussian Noise, by L. T. Remizov, pp. 270–273.

The Width of Multivibrator Spectral Lines, by B. S. Tsybakov and V. P. Iakovlev, pp. 276–280.

Simulation of Relativistic Electron Trajectories in the Magnetic Undulator, by A. N. Vystavkin, Yu. V. Anisimova and S. S. Shakhidzhanov, pp. 289–292.

Introduction: A knowledge of the trajectories of relativistic electrons in complex magnetic fields is very desirable for those cases when the fields cannot be described analytically. In particular this problem arises in the study of current flow in the magnetic undulator.

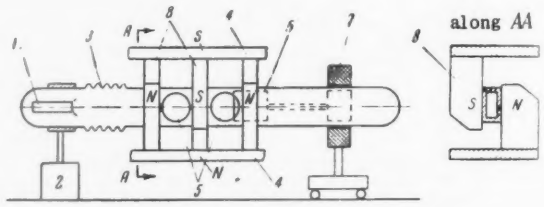

Fig. 1. A diagram of the apparatus

1 - electron gun; 2 - a mechanism for shifting the gun; 3 - bellows; 4 - magnetic bars; 5 - drift tube with hermetically sealed windows; 6 - movable collector; 7 - a device for moving an object in a vacuum; 8 - teeth which set up magnetic flux

The trajectories of relativistic electron bunches can be studied experimentally at reduced energies and hence at reduced magnetic field intensities as will be shown.

Vol. 4, no. 4, 1959.

Certain Possible Methods of Measuring the Entropy of Discrete and Continuous Message Sources, by M. P. Dolukhanov, pp. 1–10.

Introduction: Several works have been published in recent years in technical literature devoted to determining the entropy of texts written in various languages. The works of Shannon, Kupfmuller and Barnard should be mentioned here. In Shannon's work the entropy of English was determined, in Kupfmuller's work of German, while in Barnard's the results of calculating the mean value of entropy per word of English, German, French and Spanish were presented.

The determination of entropy was carried out in all cases by a method of fairly complicated calculation, at the basis of which were tables of the frequency of letters, diphthongs and triphthongs and words. To account for the deeper statistical connection between contiguous letters of text, Shannon performed an experiment employing a person as the extrapolator, with his knowledge of the grammar and structure of the English language, capable of restoring absent letters and words to an incomplete text.

Still greater difficulties arise in attempts to determine the

entropy of continuous messages, for which tables of the frequency of elements in the message are absent, while the use of individuals as extrapolators is not possible.

In the present work an attempt is made to consider the possibility of automating the process of measuring entropy both for continuous and for discrete information sources.

The Method of Isometric Transformation of Radio Lenses, by V. P. Zaborov, pp. 24–35.

Abstract: A method of constructing and calculating radio lenses with variable index of refraction is presented in this paper.

It is shown how, utilizing the results of calculation of one lens, it is possible by the use of an isometric transformation, i.e., a transformation, an invariant of which is the element of optical path in the medium with variable index of refraction, to obtain new lenses which may differ from the original in both the form of the body and the form of the phase front.

The isometric transformation represents a conformal representation of the plane of the lens profile. The index of refraction is transformed according to the formula

$$\overline{N_i} = N_i/|w^{*\prime}|$$

where $|w^{*\prime}| = |dw^*/dz^*|$ is the modulus of the derivative of the complex analytic function w^* (z^*), while $i = 1, 2, 3, \ldots$, is the index of the section of the profile plane in the limits of which N_i is a continuous function of the coordinates.

Isometric Transformation of Constant-Thickness Lenses, by V. P. Zaborov, pp. 36–47.

Abstract: A family of lenses is considered in this paper which are obtained by isometric transformation of a constant-thickness lens with variable index of refraction. The bodies of the equivalent lenses have axes of rotation normal to the plane of the lens profile and indexes of refraction which are a function of a single radius. The phase-front at the output is plane. Metal-air analogs to the lens family are found. Their mean surfaces are surfaces of revolution.

The Diffusion Equation for Statistically Inhomogeneous Wave Guide, by M. E. Gertsenshtein and V. B. Vasil'iev, pp. 74–84.

Abstract: It is shown that with regard to multiple reflections and attenuation of incident waves the probability density W for the reflection coefficient in a wave guide (long line) with a single-wave mode satisfies the diffusion equation $D\Delta W = \delta W/\delta z$, where the Laplace operator Δ is defined by a Lobachevskii-plane metric. An exact solution is found for the probability density in the case where a matched load is located at the end of the wave guide. The additional attenuation in the line due to losses of energy through reflection from random inhomogeneities is calculated. The energy incident on the thermal load is estimated after multiple reflections, constituting the cause of cross-modulation distortion of the transmitted signal.

Certain Results of Applying the Method of A Posteriori Probability to Problems of Radar System Design, by S. E. Fal'kovich, pp. 85–99.

Abstract: Results are presented of the application of a method o a posteriori probability to the problem of obtaining the circuit of a radar receiver, realizing the potentialities of the system in the reception of signals reflected from moving targets.

The Detection of Coherent Signals in the Presence of Correlated Noise, by V. D. Zubakov, pp. 100–110.

Abstract: In the development of other papers, the optimum receiver for detecting coherent radar signals during two repetition periods was considered for the cases of known and unknown Doppler shifts of frequency. A comparison was made of the optimum (two-channel) system with the sub-optimum (single-channel) system.

Noise Elimination by Pulse Signal Time Discrimination, by B. N. Mitiashev, pp. 111–125.

Abstract: The question is examined of the precision of determining the time of signal pulses, using a time discriminator in which integration of signal and noise is carried out. Formulas are obtained for the mean-square time error and for the limiting noise to signal ratio at the output of the receiver. It is shown that such a discriminator insures the theoretical attainment of limits of precision for easily realizable conditions.

Methods of Simulating Increased Power in Testing Radar Transmit-Receive Tubes, by V. E. Golant and M. Ia. Mandel'shtam, pp. 143–162.

Abstract: The article analyzes methods of simulating increased power in testing TR tubes using increased field intensity in cavity resonators. Parameters are calculated determining the functioning of installations in testing TR tubes in an ordinary cavity resonator and in a traveling-wave ring resonator. A description is given of the block diagram of the system for measurement of the parameters of the TR tubes under test in the ring resonator. Experimental results are presented.

The Theory of the Molecular Oscillator, by A. N. Oraievskii, pp. 229–236.

Abstract: The equations of the molecular oscillator are obtained, describing a stationary process arbitrarily varying in time. It is shown that the molecular oscillator may be considered as an oscillating system with inertial nonlinearity. The equations obtained are applied to the problem of synchronization of the oscillator by an external force.

The Theory of a Ferrite Microwave Amplifier, by V. V. Nikol'skii, pp. 241–243.

Vol. 4, no. 6, 1959.

Side Radiation From Ideal Paraboloid With Circular Aperture, by L. B. Tartakovskii, pp. 14–28.

Abstract: The relationship is considered between the current and aperture methods of calculating parabolic antennas and the limits of applicability of the latter are established. The magnitudes of side and back radiation of an axially symmetrical antenna are calculated, the fine structure of side radiation is investigated including the character of the directional phase diagrams.

Determination of UHF Electromagnetic Field Intensity Due to Coherent Atmospheric Scattering in the Deep Shadow Region, by L. M. Ponomarenko, pp. 29–37.

Abstract: The process of long distance UHF propagation due to coherent scattering in the troposphere is examined theoretically, using the method of Leontovich and Fok. The characteristic equation for particular values of t_3 is investigated for the case of bilinear model of the dependence of index of refraction N on height.

Numerical values for field intensity are obtained in the region of deep shadow for wave length $\lambda = 6$ m and for height of points of radiation and reception $h = h' = 300$ m.

Passage of Signal and Noise Through a Nonlinear Inertial System, by Iu. E. D'iakov, pp. 36–47.

Abstract: A general method of linearizing equations describing fluctuations in nonlinear narrow-band systems is considered for large signal to noise ratios.

Synchronization of Relaxation Oscillator by Random Voltage, by A. M. Vasil'iev, pp. 47–57.

Introduction: The action of random voltage on an electronic relay has been considered in several works. In the present work, for certain special assumptions, there are considered the effects of random voltage on an electronic relay in the self-oscillatory regime, i.e., on a relaxation oscillator. In connection with the wide use of these relays in television and radar engineering, this problem has substantial practical interest.

The following order of discussion has been adopted for the present article. First is given a mathematical statement of the problem. Then equations are derived for the functions giving the necessary information on the random voltage at the output of the relaxation oscillator. These equations are solved for the case where stationary random voltages act at the inputs of the oscillator and, in a particular case, nonstationary random voltages.

Lens Achromatization and Scale Correction in Reflex Electron Microscopes, by G. V. Der-Shvarts and Iu. M. Kushnir, pp. 129–137.

Abstract: The results of partial achromatization of a reflex electron microscope and the correction of scale distortions connected with low observation angles are considered.

Conclusions: It is demonstrated that the optical system of a reflex electron microscope may be partially achromatized. After such achromatization the microscope is capable of a resolution of 500–600 Å. The scale distortion characteristic for a reflex microscope may also be partially eliminated if an additional

element be introduced into the microscope column—a cylindrical lens scale corrector. With its aid the ratio of scales $M'_{\perp}/M'_{\parallel}$ may be raised to 3–2 at an angle of observation $\theta_2 = 7–8°$. Since such a reflex microscope has four times better resolution than a good optical metallurgical microscope, the given instrument may be successfully used for direct metallographic studies.

Energy and Emission Angle Distributions of Photoelectrons From Compound Cathodes, by N. S. Khlebnikov and A. E. Melamid, pp. 138–151.

Abstract: The work describes and evaluates experiments studying the energy (for Sb-Cs and Cs-O-Ag cathodes) and emission angle (for Sb-Cs cathode) distributions of photoelectrons.

Electrical Properties of TiO_2 Rectifiers, by T. I. Komolova and D. N. Nasledov, pp. 175–182.

Abstract: A method is described for obtaining junction rectifiers on the basis of semiconductivity of titanium dioxide. The rectifiers have satisfactory parameters and may operate at elevated temperatures. The volt-ampere characteristics of the elements are presented; the dependence of rectifier parameters on temperature (in the interval of temperatures from 20 to 300 C) is described.

Conclusions:

1 The rectifiers obtained had satisfactory electrical parameters. They may function at reverse voltages up to 10 v and permit relatively large current densities in the forward direction.

2 The rectifiers may work satisfactorily up to 200 C. However, this is evidently far from the upper limit of working temperatures.

3 The dependence of capacitance and rectifier resistance of bias voltage and temperature is similar to that occurring in the case of selenium rectifiers and is found in qualitative agreement with theory. The high capacitance of the rectifiers is due to the high dielectric constant of titanium dioxide.

4 Above 200 C there are observed certain irreversible changes which are possibly connected with diffusion processes.

Transistor Integrating Amplifier, by A. A. Groshev, pp. 183–194.

Abstract: Expressions are presented for the transfer factor and time constant of a transistor integrator. The effects of transistor input resistance on the precision of integration are considered.

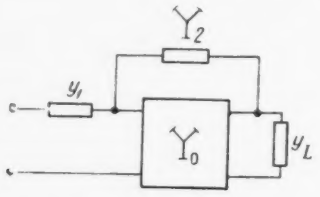

Fig. 1. Circuit of integrating amplifier in general form

A circuit is given for an integrating amplifier with input impedance greater than the collector resistance.

The Sensitivity of a Radiometer With Low Noise Level (in the Quantum Region), by N. V. Karlov and B. M. Chikhachev, pp. 195–200.

Abstract: The question is considered of the limiting accuracy of measurement of the internal noise level of a radiometer when the noise temperature is close in order of magnitude to $\hbar\omega/k$ and the noise power is determined by the quantum Nyquist formula. Expressions are obtained for the radiometer sensitivity in this case. The optimum internal-noise temperature is found for which the sensitivity of the radiometer is maximum. These formulas have significance for the use of quantum-mechanical amplifiers in the band of centimeter wave lengths.

Sensitivity of a Radiotelescope at Low Input Noise Levels, by N. V. Karlov and B. M. Chikhachev, pp. 201–207.

Abstract: The question is considered of the limiting sensitivity of a radiotelescope in measuring the effective temperature of cosmic sources of radiation in the case of low input noise levels. It is shown that the sensitivity of the radio telescope is essentially determined by quantum effects even in the case where the temperature of its input noise substantially exceeds $\hbar\omega/k$.

Representation of a Multidimensional Normal Probability Density in the Form of a Multiple Series, by A. I. Glazkov, pp. 211–212.

Introduction: The solution of certain radio engineering problems using the theory of random functions leads to n-dimensional integrals with a kernel in the form of multidimensional probability densities, not obtainable in finite form. In this case difficulty arises mainly due to the impossibility of separating the variables of integration. Representation of the probability density in the form of a series in powers of correlation functions gives the possibility of expressing the solutions of such problems by means of series.

Thermionic Properties of Oxides of Group III and IV Metals, by B. V. Bondarenko and B. M. Tsarev, pp. 213–216.

Abstract: Oxides of metals of groups III and IV of the periodic table of elements: Sc_2O_3, Y_2O_3, La_2O_3, TiO_2, ZrO_2, HfO_2 are fairly refractory compounds and may present interest along with thorium oxide as materials for thermionic cathodes operating at elevated temperatures. However the thermionic properties of these materials have been insufficiently studied. Even without regard to the fact that relatively many works have been devoted to the thermionic properties of ThO_2, recently there have appeared new quantitative data for the constants φ_0 and A in the equation of emission $j = AT^2 \exp(-e\varphi_0/kt)$, different from those published previously. We have carried out an investigation of the thermionic properties of these oxides and thorium oxide over a fairly wide interval of temperature, by applying them to tungsten tape and recording the data to construct the Richardson lines.

New Method of Shock Excitation of Tuned Circuit Oscillations, by G. N. Berestovskii, pp. 217–219.

Periodic Nonstationary Processes, by L. I. Gudzenko, pp. 220–224.

Introduction: A series of theoretical works has recently appeared on radio physics, in which are considered periodic nonstationary processes, i.e., random functions of time with periodically varying mean values and correlation functions. In the present work are considered certain general properties of these processes.

Transforming Thermal Into Electrical Energy by Utilization of Thermionic Emission, by N. D. Morgulis and A. G. Naumovets, pp. 225–227.

Vol. 5, no. 1, 1960.

The Oscillations of a Small Gyrotropic Sphere in the Field of a Plane Wave, by A. A. Pistol'kors and Siu Yen-Sheng, pp. 1–17.

Abstract: The excitation of incident plane waves of electromagnetic oscillations in a gyrotropic sphere of small radius and the diffraction phenomena entailing it are investigated. In calculating the magnetic field, the solutions of Walker are used as a basis. The dependence of the amplitude and the resonance curve of the oscillations upon various factors is studied. The value obtained for the magnetic induction is employed in determining the electric field. As a result of the gyrotropic character of the sphere a transformation of the types of oscillations appears to be possible: The Debye potential of the diffracted waves might differ in its type from the potential of the energizing waves.

An Investigation of the Natural Frequency Spectrum of a Resonator Containing a Magnetized Ferrite, by A. L. Mikaelian, A. K. Stoliarov and A. A. Vasil'yev, pp. 33–54.

Abstract: An exact solution is obtained for the problem of the natural frequencies of a cylindrical resonator (with oscillations of the form $TM_{q,n,0}$), containing an axially magnetized ferrite cylinder of arbitrary radius. The natural frequencies are determined and the structure of the field is investigated.

The analogous problem is solved for the case where the side walls of the resonator are removed. It is shown that for cylinders of infinitesimally small radius the results obtained agree with the corresponding results of Walker determined in the quasi-static approximation.

Nonlinear Theory for Ferrite Oscillators, by A. L. Mikaelyan, pp. 66–83.

Abstract: In this article questions of nonlinear theory, which permits us to establish not only conditions for the excitation of parametric oscillations, but also to calculate the amplitude of the oscillations in the resultant working conditions, are discussed. The results are adduced of the calculation of the resonance curves for a generator of the electromagnetic type, and a comparison is made of the phenomena in the case under consideration with the well studied phenomena in mechanics and radio technics.

Certain Questions of the Theory of Parametric Amplifiers, by V. I. Subkov and Ia. A. Monosov, pp. 107–127.

Abstract: The natural frequencies (frequencies of parametric coupling) and resonance curves of parametrically coupled circuits are investigated.

Curves of variation of natural frequencies (frequencies of parametric coupling) are calculated with detuning of one of the coupled circuits and the detuning region is found in which parametric excitation of oscillations occurs.

The question of the resistance introduced into the pumping circuit is considered and conditions are found for which it has maximum value.

Ferrite Microwave Mixers, by A. L. Mikaelian and V. Ia. Anton'iants, pp. 128–149.

Abstract: The effect of electromagnetic oscillations at two nearby frequencies on a small premagnetized ferrite sample is considered.

The amplitude in the load circuit at the difference, equal to 30 Mc per sec, is calculated and the results of measurement of this quantity presented for monocrystalline and polycrystalline ferrite spheres of diameters from 0.5 to 3 mm.

Measurements and calculations of difference frequency voltage for monocrystalline samples agree fairly well with each other.

For polycrystalline samples greater differences were found between calculation and experiment. An attempt is made to explain these differences.

In addition results are presented of a study on cylindrical ferrite samples of various types. The smallest conversion loss obtained in the experiments was 58 db at a local-oscillator power of 50 mw.

Experimental Study of Nonlinear Microwave Phenomena in Ferrites, by Ia. A. Monosov and A. V. Vashkovskii, pp. 150–167.

Abstract: The results of an experimental study of additional absorption in ferrites at high power level in the 3-cm band are described and compared with theory. It is established that the additional absorption in the subresonant magnetization field region consists of individual absorption peaks and an interpretation of this phenomenon is given.

The dependence of the threshold microwave field levels, at which nonlinear phenomena arise, on the saturation magnetization, shape and dimensions of a ferrite sample are given.

Ferromagnetic Amplifier Theory Based on the Perturbation Principle, by V. V. Nikol'skii, pp. 203–215.

Introduction: Different forms of Suhl's ferromagnetic amplifier are resonant systems. A traveling-wave ferromagnetic amplifier has been proposed. In all these systems increase of amplitude at the frequency ω occurs by virtue of a process in magnetized ferrite under the action of a field at a higher frequency ω^s, supplying energy (we shall employ the term "supply field"). The effect of this field, specifically changing the magnetic susceptibility of the ferrite, is conveniently considered as a perturbation of the system, the result of which is a shift in its characteristic complex value, i.e., the natural frequency of the wave number—depending on whether we are concerned with a resonator or a wave guide. (By the term "wave guide" we shall understand in the entire article some guide system.) It is important that with this approach the unperturbed system is the amplifier with magnetized ferrite. In view of this there is no essential difference in analysis between "electromagnetic," "magnetostatic" and "semistatic" forms; any arbitrary oscillations or waves are described by the characteristic functions of the unperturbed system.

The perturbation principle has been applied to the theory of the ferromagnetic amplifier.[3] The present work is in further development of this question.

[3] V. V. Nikol'skii, *Radiotekhnika i Elektronika*, vol. 4, no. 4, 1959, p. 726. (*Radio Engineering and Electronics.*)

RUSSIAN JOURNAL OF PHYSICAL CHEMISTRY (Zhurnal Fizicheskoi Khimii). Published by The Chemical Society, London.

Vol. 33, no. 8, Aug. 1959.

Retardation of the Motion of Gas Bubbles by Surface-Active Substances at Moderate Reynolds Numbers, by A. I. Fedosov, pp. 109–111.

Summary:

1 The motion of a bubble is examined at moderate Reynolds numbers in the presence of a surface-active substance.

2 The surface of the bubble over which the tangential motion can be regarded as considerably retarded has been calculated approximately.

3 The concentrations of surface-active agents have been calculated at which (a) the surface of the bubble can be regarded as completely free from surface-active substances and (b) the bubble moves as if it were a solid sphere.

Diffusion in Liquefied Gases, by V. G. Manzhelii and B. I. Verkin, pp. 147–148.

Introduction: Study of diffusion in liquids is of interest in connection with its sensitivity to structure. Since an existing theory is claimed to apply to simple liquids alone, it is appropriate to check it by studying diffusion in liquid metals and liquefied gases.

The purpose of the present work was to work out a simple and reliable method for studying diffusion in liquefied gases. We used the principle of the gas phase method, first suggested by Ryskin and Zhurkov and later applied to liquids. It is preferable to the capillary method, which was employed in the only paper known to us on diffusion in liquefied gases, because it enables convection to be controlled and a continuous record of the relation between diffusion coefficient and concentration to be made.

The methane-propylene system was chosen for investigation first, because of the large difference in the vapor pressures of the components, which makes the experiment easier to carry out and increases the accuracy of the measurements made by the gas phase method. The considerable solubility of methane in propylene makes it possible to study the concentration dependence of the diffusion coefficient. Another reason for studying this system is the comparative simplicity of the molecules.

Indirect Method of Estimating the Reaction Zone Width of a Detonation Wave, by L. N. Stesik and L. N. Akimova, pp. 148–151.

Introduction: We have derived zone widths for certain explosives from the relation of detonation rate to radius of charge. As material is ejected from the reaction zone, the zone width must determine the relation of detonation rate to charge radius. Hence we may find the zone width from the relation of rate to radius. The zone width is defined as the distance from the leading edge of the detonation front to the Jouget point.

Summary:

1 A curve has been drawn up which gives the reaction zone width in a detonation wave from the relation of detonation rate to radius of charge.

2 The zone widths and reaction times have been established for various densities of trotyl, hexogene, and their mixtures.

Thermodynamic Properties of Liquid Metallic Solutions. The Potassium-Thallium System, by M. F. Lantratov and E. V. Tsarenko, pp. 161–164.

Summary:

1 Electromotive forces of the concentration cells K-glass and K–Tl alloy were measured within the range 475–525 deg for solutions containing 12.6–95 at. % Tl.

2 The values of the activity, activity coefficient, partial molar free energy and partial molar excess free energy were calculated for K and Tl at 525 deg. The integral values of the molar free energy, molar excess free energy, molar entropy of mixing, molar excess entropy of mixing and molar heat of mixing were also evaluated.

3 Considerable negative deviations for ideal laws are observed in the system, increasing as the temperature decreases. This behavior is related to the strong interaction between the solution components.

4 The minima in the integral curves lie at $N_K = 0.4$. The

displacement of the minimum away from the composition of the most stable compound is apparently related to the fact that, as well as KTl, the liquid alloy also contains compounds more abundant in Tl.

The Homogeneous-Heterogeneous Combustion of Carbon Monoxide in Narrow Ducts, by L. S. Solov'eva, pp. 171–173.

Summary:

1 Theoretical analysis of the conditions for homogeneous-heterogeneous processes by means of averaged combustion equations shows that these processes should be studied in narrow tubes of materials with different catalytic activities.

2 A method has been developed for the study of the homogeneous-heterogeneous combustion of gases in 2-mm diameter cylindrical ducts under isothermal conductions.

3 Carbon monoxide–air mixtures have been studied at 500–900 deg in a quartz duct at flow rates of 1–34 m per sec and initial carbon monoxide concentrations of 2–5 vol %.

4 The quantities k_0 and E characterizing the overall reaction of homogeneous combustion of carbon monoxide in the duct at 500–900 deg have been determined, and the activation energies obtained for temperatures above 800 deg have been compared with data of other workers. These data are necessary for further study of the heterogeneous process.

Variation With Temperature of the Width of the Components of the Hyperfine Structure of Electron Paramagnetic Resonance Spectra, by N. N. Bubnov and V. M. Chibrikin, pp. 206–208.

Introduction: In order to clarify the detailed mechanism of the broadening of the components of the hyperfine structure, we have studied the effect of temperature on the electron paramagnetic resonance spectra of dibenzenechromium $(C_6H_6)_2Cr^+$(I) and bisdiphenylchromium $[(C_6H_5)_2]Cr^+$(II). The experiments were carried out with concentrated solutions ($c > 0.02$ M) with a spectrometer having a high frequency modulation of the magnetic field. As solvents we used ethyl alcohol, pyridine and acetone.

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Thermodynamic Properties of Germanium-Zinc Alloys, by G. F. Voronin and A. M. Evseev, pp. 268–270.

Introduction: Thanks to its electrophysical properties, germanium finds wide technical application. Up to the present, the thermodynamic properties of alloys of this element have not been described in the literature. An investigation of these would be useful both for the development of general ideas on the nature of metallic solutions and also, for example, in working out methods for purifying germanium.

We have investigated the thermodynamic properties of liquid and solid germanium-zinc alloys over the temperature range 342–466 deg by measuring the pressure of saturated zinc vapor. We have previously described a simple and reliable modification of this method for determining the activity of the volatile component of an alloy, based on the use of a continuously weighing balance.

Summary:

1 The thermodynamic properties of germanium-zinc alloys have been investigated. The system shows large negative deviations from ideal behavior.

2 It is concluded that germanium, in forming solutions with zinc, strongly distorts the zinc structure in both solid and liquid states.

Experimental Determination of the Attenuation Coefficients of γ Radiation in Cylindrical Emitters, by N. G. Smirnov and K. A. Uspenskii, pp. 273–275.

Introduction: In physico-chemical laboratory practice, use is made of cylindrical liquid γ emitters; the calculation of the dose power from these is difficult, since there are no data for the self-scattering of γ quanta in emitters. The coefficient for self scattering has been taken as 1–1.5, depending on the solid angle from which the source is viewed.

An attempt has been made to derive a mathematical expression for the attenuation coefficient in cylindrical emitters, taking account of self-absorption and self-scattering. The author takes account of the self-scattering by an analytical presentation of the accumulation factors published in a number of foreign sources. A simple expression for the accumulation factor has been given by Kalinin and Vertebnyi, who allowed for the scattered radiation using the factor $1 + \alpha\mu l$ (where α is a constant, μ the γ quantum absorption coefficient, and l the thickness of the material) and obtained a simple integral formula for calculating the dose power of γ radiation from three-dimensional emitters.

The present work reports the experimental determination of the attenuation coefficients in cylindrical emitters.

Summary:

1 The attenuation coefficients in cylindrical emitters of different radii, filled with Co^{60} salt solution, have been determined experimentally.

2 The attenuation coefficients are independent of the distance from the axis of the cylindrical emitter.

3 The self-scattering is considerable and must be taken into account when calculating the yield of γ radiation from cylindrical emitters.

(Continued from page 96)

is Maxwellian when seen by an observer moving with the mean gas velocity; that is, the velocity $\vec{} - \vec{U}$, where $\vec{u}$, the molecular velocity observed from the (fixed) surface, is distributed according to the law

$$f_i = \frac{n_i}{(2\pi k T_\infty/m_i)^{3/2}} \exp - \left[\frac{(u_x - U_x)^2 + (u_y - U_y)^2 + (u_z - U_z)^2}{2kT_\infty/m_i}\right] \quad [1]$$

where

f_i = number of particles of ith species per unit volume with velocities in the range $\vec{u} - \vec{U}$ to $(\vec{u} - \vec{U}) + d\vec{u}$
n_i = concentration of ith species, number/unit volume
k = Boltzmann's constant
T_∞ = ambient gas temperature
m_i = mass of particle of ith species
u_x, u_y, u_z = molecular (or particle) velocity components in x, y, z directions as seen from the fixed surface

The flux of the ith constituent at the surface is given by

$$J_i = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{0}^{+\infty} f_i u_x du_x du_y du_z \quad [2]$$

The integration is performed over all values of u_y and u_z, but only over those particles with positive values of u_x at the surface. Evaluating the integral gives (2,3)

$$J_i = (n_i C_i/2\pi^{1/2})[e^{-\alpha^2} + \pi^{1/2}\alpha(1 + \text{erf } \alpha)] \quad [3]$$

where

$C_i = (2kT_\infty/m_i)^{1/2}$
$\alpha = S \sin \theta$
S = "molecular speed ratio" = U/C_i
$U = |\vec{U}|$
θ = angle between $\vec{U}$ and the surface

The flux over the rear side of the plate is determined by integrating over all values of u_y and u_z, but with the limits on u_x taken from $u_x = 0$ to $u_x = -\infty$. The result is (2,3)

$$J'_i = (n_i C_i/2\pi^{1/2})[e^{-\alpha^2} - \pi^{1/2}\alpha(1 - \text{erf } \alpha)] \quad [4]$$

This is the expression which would be obtained by setting $\alpha = -\alpha$ in Equation [3].

When the gas has no mean motion, $\vec{U}$ is equal to zero, and these expressions reduce to the usual value for the molecular flux obtained from kinetic theory

$$J_{i0} = n_i(kT_\infty/2\pi m_i)^{1/2} \quad [5]$$

Impaction on Ribbons, Spheres and Cylinders

Using Equations [3 and 4], it is possible to calculate the efficiency of impaction of suspended particles on a body moving through the gas. The impaction efficiency, defined as in continuum flow, is the fraction of the particles present in the gas swept out by the body which strike its surface. In this note, expressions for the efficiency will be derived for the flat ribbon set at an angle to the flow, the sphere and the cylinder placed normal to the flow (Figs. 1). For these three bodies, each element of area exposed to a velocity U has an equivalent element exposed to a velocity $-U$. Hence the total flux for each differential area, front and rear, is

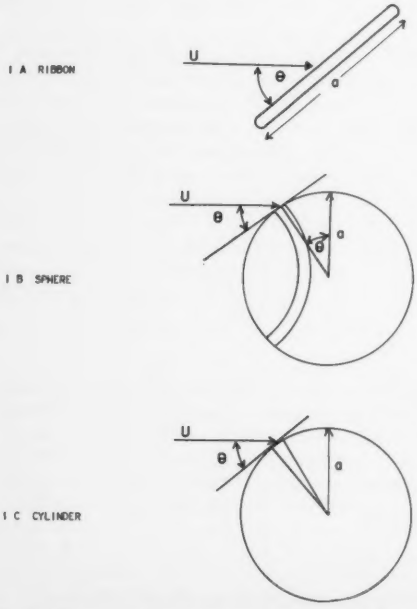
$$J_i + J'_i = (n_i C_i/\pi^{1/2})(e^{-\alpha^2} + \pi^{1/2} \text{ erf } \alpha) \quad [6]$$

The efficiency for the ribbon can be obtained immediately

$$\eta = \frac{J_i + J'_i}{n_i U a \sin \theta} = \frac{1}{S \sin \theta} \times \left[\frac{e^{-S^2 \sin^2 \theta}}{\pi^{1/2}} + S \sin \theta \, (\text{erf } S \sin \theta)\right] \quad [7]$$

In the case of the sphere, it is convenient to choose as a differential area the strip $2\pi a^2 \cos \theta d\theta$ shown in Fig. 1b. The efficiency is then

$$\eta = \int_0^{\pi/2} 2\pi a^2 (J_i + J'_i) \cos \theta d\theta / n_i U \pi a^2$$
$$= \frac{2}{\pi^{1/2} S^2}\left[\int_0^S (e^{-\alpha^2} + \pi^{1/2}\alpha \text{ erf } \alpha) d\alpha\right]$$
$$= 1 + \frac{1}{2S^2} \text{erf } S + \frac{1}{S} i \text{ erfc } S \quad [8]$$



Figs. 1 Impacting surfaces showing elemental areas

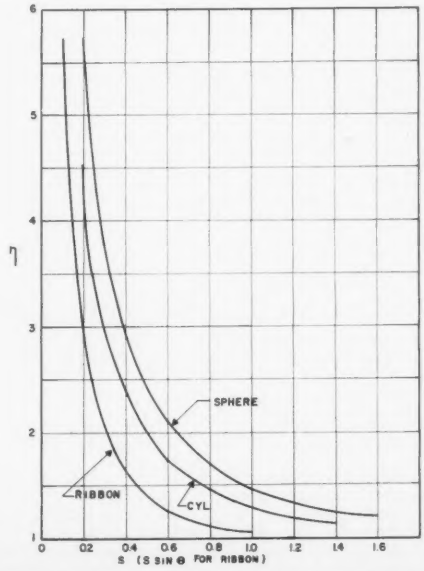


Fig. 2 Impaction efficiencies for free molecule flow

In the case of the cylinder, the efficiency is given by (see Fig. 1c)

$$\eta = 2\int_0^{\pi/2} a(J_i + J'_i)d\theta/2aUn_i \qquad [9]$$

The integrals $\int_0^{\pi/2} J_i d\theta$ and $\int_0^{\pi/2} J'_i d\theta$ have been evaluated by Stalder, Goodwin and Creager (4). The result is

$$\eta = (Z_1 + Z_2)/2\pi^{1/2}S \qquad [10]$$

where

$Z_1 = \pi e^{-S^2/2} I_0(S^2/2)$
$Z_2 = \pi S^2 e^{-S^2/2}[I_0(S^2/2) + I_1(S^2/2)]$
I_0 = modified Bessel function of first kind and zero order
I_1 = modified Bessel function of first kind and first order

The impaction efficiencies are plotted in Fig. 2. In general, for free molecule flow the impaction efficiency is greater than unity and approaches one as the body velocity (and speed ratio) approach infinity. The efficiency is independent of the dimensions of the impacting surface.

As an example of the application of these results, one may consider the impaction of molecular clusters 10 Å in diameter on a spherical body traveling at an altitude of 75 miles. At this altitude, the air temperature is 490 K (5). Assuming a cluster density of unity, the mass is about 0.5 $(10)^{-21}$ gm. The average speed C_i is

$$C_i = \left(\frac{2kT_\infty}{m_i}\right)^{1/2} = \left[\frac{2(1.38)(10)^{-16}\,490}{0.5\,(10)^{-21}}\right]^{1/2} = 538 \text{ fps} \qquad [11]$$

From Fig. 2, the impaction efficiency for a collector traveling at 269 fps is 2.45, at 538 fps, 1.47, and at 807 fps, 1.20.

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Theory of Motion of a Near Earth Satellite[1]

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FIRST- and second-order[4] solutions of the equations of motion (1–3)[5] of a near Earth satellite have been obtained (9). These solutions are completely general without restrictions on eccentricity or orbital inclination. Though the traditional orbital plane of astronomy is employed, a non-elliptical orbit in this plane displays the basic geometry more readily than would the traditional osculating ellipse.

To first order, the radial distance is given by

$$\frac{1}{r} = \frac{1}{r_0}\,[1 + e\cos(\beta - \omega) - Jv]$$

where r_0 and e are arbitrary constants

$$\frac{d\omega}{d\beta} = \frac{J}{2}\left(\frac{R}{r_0}\right)^2 (5\cos^2 i_0 - 1)$$

$$v = \left(\frac{R}{r_0}\right)^2 \sin^2 i_0 \left[\frac{1}{3}\left(\frac{1}{2} + \frac{1}{12}e^2\right)\cos 2\beta + \frac{1}{12}e\cos(3\beta - \omega) + \frac{e^2}{72}\cos(4\beta - 2\beta) + \frac{e^2}{8}\cos 2\omega\right] +$$
$$\frac{e^2}{6}\left(\frac{R}{r_0}\right)^2 \left(1 - \frac{3}{2}\sin^2 i_0\right)\cos(2\beta - 2\omega)$$

and R is the equatorial radius of Earth. In these expressions, β is the central angle in the orbital plane measured from the mean line of nodes, and i is the angle of incidence of the orbital plane. See Fig. 1. To first order, the latter is given by

$$i = i_0 + \frac{J}{4}\left(\frac{R}{r_0}\right)^2 \sin 2i_0 \left[\cos 2\beta + \frac{e}{3}\cos(3\beta - \omega) + e\cos(\beta + \omega)\right]$$

where i_0 is an arbitrary constant. The instantaneous line of nodes regresses (at a variable rate), and to first order is given by

$$\Omega = J\left(\frac{R}{r_0}\right)^2 \cos i_0 \left[\beta - \frac{1}{2}\sin 2\beta + e\sin(\beta - \omega) - \frac{1}{6}e\sin(3\beta - \omega) - \frac{1}{2}e\sin(\beta + \omega)\right]$$

where the secular term defines the mean line of nodes.

The second-order solution introduces long period variations in r_0, e, $d\omega/d\beta$ and i_0 and a nonlinear coupling between the latter two which has not been recognized in earlier second-order theories (4–8). When this nonlinearity is incorporated into the analytical development, the true characteristics of the apsidal motion are revealed; they are embodied in a modified pendulum equation. The familiar, nearly monotonic rotation of the line of apsides turns out to be the analog of a pendulum swinging completely about its pinion, i.e., a pendulum going "over the top," whereas the hitherto undisclosed apsidal motion near the critical angle of incidence of the orbital plane 63.4 deg is the analog of an "oscillating" pendulum. In the latter case, the librations center about points of maximum latitude in either hemisphere.

When e is of nominal size ($e \gg J$), the orbit is essentially

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[4] The potential contains, in addition to the fundamental, the second and fourth spherical harmonics. The second is considered to be small of first order (in the oblateness parameter J) and the fourth is considered to be small of second order.

[5] Numbers in parentheses indicate References at end of paper.

an ellipse rotating slowly in the orbital plane (the apsidal motion as determined by the variations in ω). The mean radius r_0, the mean eccentricity e and the mean rate of rotation of perigee $d\omega/d\beta$, each experiences long-period perturbations. The traditional *osculating* values of these quantities (5) would, in addition, reflect the short-period perturbations which here have been relegated to the additive term Jv in $1/r$. This form, in which the long- and short-period perturbations are separately displayed, serves to clarify the total description of the satellite motion.

When e is small (of order J), the total orbit bears little resemblance to an ellipse. In fact, Jv is then very nearly given by the single term

$$\frac{1}{6} J \left(\frac{R}{r_0}\right)^2 \sin^2 i_0 \cos 2\beta$$

which becomes dominant as $e \to 0$. Since this term is of double frequency, more than one perigee (i.e., local minimum[6] of r) may occur in a single revolution of the satellite. In addition, this term does not share in the "apsidal motion" associated with the fundamental, and so the orbit shape is continually subjected to major variations. It is clear that traditional orbital concepts based upon the elements of an osculating ellipse become meaningless for these nearly circular orbits.

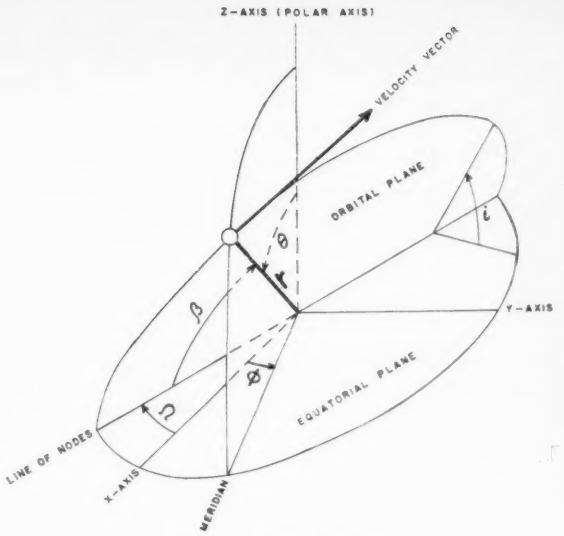

Fig. 1 Orbital geometry

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[6] The radial distance r may exhibit as many as three minima per revolution.

Densities of the Upper Atmosphere Derived From Discoverer Satellites

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PRELIMINARY analysis, made in December 1959, of the Discoverer satellite orbits has provided additional data for the determination of the density of the upper atmosphere between 100 and 150 statute miles. These data have confirmed the density model currently being used in the Lockheed MSD computer program for ephemeris predictions, and agrees very well with the density in the 1959 ARDC Atmosphere Model.

Analysis

The near polar orbits of the Discoverer satellites are one of the features that make it particularly attractive in the calculations of the atmospheric density. The high inclination has allowed the argument of perigee to proceed from the equatorial belt to the highest northern latitudes and, in some cases, has also moved to the night side of the orbital plane. Although the Discoverer vehicle is attitude stabilized for the first portion of its life, the recovery sequence produces an abrupt change in the orbit and the vehicle assumes a new drag parameter. It is from this point that the atmospheric drag calculations have been made. In the determination of the density from observations of the motion of a satellite, the basic parameters are the drag parameter $C_DA/2m$ and either the vehicle acceleration or its total lifetime. Although the total lifetime is usually known within a revolution or two the acceleration is more difficult to determine with the limited number of tracking stations available to the computer center. When the vehicle's orbit has completely decayed, the total lifetime is then known and can be related to the density as indicated in (1)[2]; the basic equation is repeated

$$\rho_{h_p} = \frac{m f(B)}{C_D A \, a^2 2\pi k N_T}$$

At any select time during the vehicle lifetime the total number of revolutions N_T can be considered as the total remaining revolutions, and the resultant density is related to the orbital parameters at that select time.

The density is also related to the acceleration using Sterne's equation (2) which is also repeated

$$\rho_{h_p} = \frac{m \dot{P} \, C^{1/2}}{C_D A \, a \, f(c,e)}$$

For this analysis the Discoverer 1103-A computer program (3) was used to calculate the orbit parameters for Discoverers VI, VII and VIII using all available tracking data. Initial orbit parameters for all Discoverer vehicles are tabulated.

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[2] Numbers in parentheses indicate References at end of paper.

Table 1 Initial orbit parameters for Discoverer vehicles

Discoverer no.	Perigee ht., statute miles	Incl., deg	Period, min	Lifetime (actual), orbit revs.	Lifetime (predicted) orbit revs.	Lifetime prediction error
II	158	89.89	90.4	205	213	+4%
V	134	80.01	94.2	710	862	+21%
VI	137	84.26	95.3	965	1174	+22%
VII	102	81.66	94.7	297	285	−4%
VIII	115	80.69	103.7	1595	1311	−18%

The 1103-A is a Remington Rand digital, alpha-numerical, internal storage computer operating on a program containing: A gravity model using only the first expansion term J equal to 1.637×10^{-3}; a density model which, for altitudes above 486,000 ft, is computed by the expression

$$\rho_h = 0.40174929 \times 10^{-11} \left(\frac{h}{486{,}000}\right)^{-5.55} \frac{\text{slugs}}{\text{ft}^3}$$

and an Earth model whose eccentricity is e equal to 0.0819918898. The semimajor axis a equals 2.0926041×10^7 ft and the semiminor axis b equals 2.0855583×10^7 ft.

This density model was established at the Lockheed Missiles and Space Division in December 1958 based on calculations of the acceleration of 1958 Zeta together with the revised data from previous satellites.

In order to proceed through the lifetime of the vehicle, the tracking data were selected in overlapping groups proceeding from the recovery position. This grouping was necessary to provide a means of updating the input data during periods of little or no tracking information. In these calculations it was assumed that the vehicle was tumbling in a random fashion with the orbit decay due entirely to aerodynamic drag occurring at perigee. The area and weight of the tumbling vehicle were redetermined after examination of the telemetry obtained during the recovery sequence.

The major portion of the tracking data after the recovery sequence was obtained from the National Space Surveillance Control Center (Space Track). Both the orbit acceleration as determined by the computer and the total lifetime of Discoverers VI and VII were used to calculate the density, but only the acceleration of Discoverer VIII was used, since at the time of this analysis it was still in orbit. As would be expected the densities obtained using each parameter were quite similar, and only the values as given by the accelerations are presented here.

Conclusion

The results of this study are presented in Fig. 1 which also shows the density from the ARDC Atmosphere Model, 1959, and data from previous satellites as reported elsewhere. Considerable effort was made in an attempt to correlate the density variations with the change in the argument of perigee, but the time interval between successive tracking data within the grouping was great enough to mask this effect. This still leaves unanswered the finer points of latitudinal, diurnal and seasonal variations of atmospheric density, but succeeds in confirming previous estimates of the variation in density with altitude within the operating limits of the vehicle.

Comments

If it were possible to obtain more continuous observations of satellites with near polar orbits, considerably more scientific information could be obtained at no expense to the primary mission of the vehicle. This is well illustrated in the orbital behavior of Discoverer VIII (1959 Lambda). At the time of this writing observational data indicate that, after approximately 110 days, Discoverer VIII has made its final revolution. During its lifetime the perigee has made one complete revolution of the orbital plane covering the latitudes from 80 deg north to 80 deg south, while the orbital plane has also rotated approximately 110 deg. This combined rotation has exposed a single satellite to every environment that is possible for a close Earth satellite.

Acknowledgments

I am very grateful for the assistance of Mr. R. L. Gault in the density calculations and of Mr. J. B. Reed in the computer reduction of the tracking data. Both are at Lockheed Missiles and Space Division.

Nomenclature

- ρ = atmospheric density
- $\dot{P}$ = acceleration
- m = mass
- A = satellite area projected on a plane normal to the direction of motion
- C_D = dimensionless aerodynamic drag coefficient

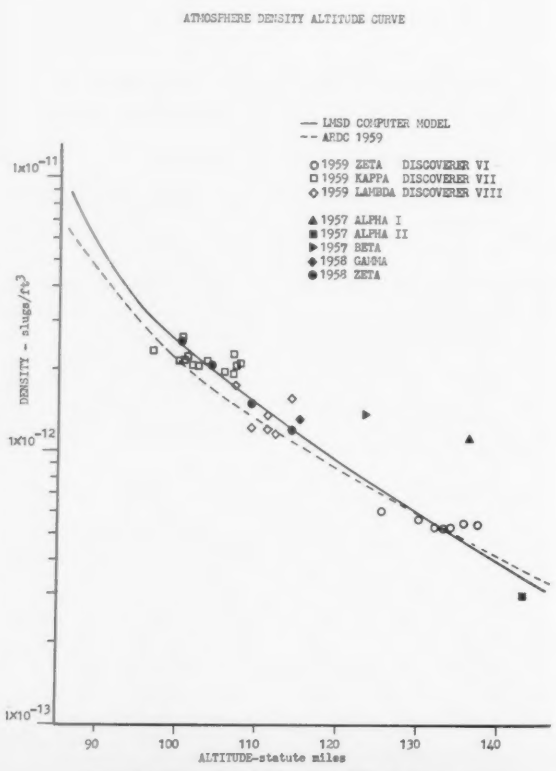

Fig. 1 Atmospheric density altitude curve

a = semimajor axis
a_0 = initial semimajor axis
c = kae
e = eccentricity
k = logarithmic gradient density near perigee

$$f(c, e) = 1 + 2e + \frac{3e^2}{2} + \frac{1 - 6e - 10.5e^2}{8c} + \frac{9 + 30e + 85.5e^2}{128c}$$

B = $\frac{1}{2}kae$
$f(B)$ = $Be^B/2I'_0(B)$
$I'_0(B)$ = modified Bessel function

h = height
h_p = perigee height

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1 Breakwell, J. V. and Koehler, L., "Elliptical Orbit Lifetimes," Lockheed Missiles and Space Division Report no. 2463, May 23, 1958, Lockheed Aircraft Corp., Sunnyvale, Calif.

2 Sterne, T. E., "The Density of the Upper Atmosphere," IGY Satellite Report Series, no. 3, May 1, 1958.

3 "Discoverer Tracking, Acquisition and Control Program for the 1103-A Digital Computer," Digital Programming Department, Lockheed Missiles and Space Division Report no. 445288, July 30, 1959, Lockheed Aircraft Corp., Sunnyvale, Calif.

Solid Propellant Source of Cesium Plasma[1]

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AN INVESTIGATION is made of the combustion characteristics and ion and electron density obtainable by burning mixtures of $CsClO_4$:Al and $CsNO_3$:Al. Thermodynamic equilibrium calculations made with an IBM-704 computer for stoichiometric $CsNO_3$:Al and $CsClO_4$:Al are presented in Tables 1 and 2, respectively, which give the temperatures and percentages of the significant ion species for three-pressure and two-pressure expansions.

It is seen that expansion causes an increase in the equilibrium ion concentrations for these conditions. It should be noted that the absolute level of the total ion and electron concentration is quite high, e.g., 9.8×10^{16} per cc at 14.7 psi and 3.0×10^{18} per cc at 1000 psi for $CsNO_3$:Al. Results of calculations with stoichiometric $CsClO_4$:Al are similar, except that the most abundant negative species is Cl^- rather than free electrons.

The mixtures were found to be burnable when prepared as strands (4 × 4 × 38 mm) formed by compression at various forming pressures up to 125,000 psi. Observations of the burning with a high speed camera, and burning rate measurements, were made at various ambient nitrogen pressures up to 2000 psi. The $CsNO_3$:Al strands would not burn below about 200 psi, the burning rate at 300 psi decreasing from 0.2 ips at a forming pressure of 3300 psi to 0.12 ips at a forming pressure of 125,000 psi. Burning rates increased with increasing nitrogen pressure, the pressure exponent well below unity except at the lowest forming pressure (3300 psi). It was found that a weakly tamped stoichiometric mixture of $CsNO_3$:Al powders in a tube would burn at atmospheric pressure. The $CsClO_4$:Al strands would burn at atmospheric pressure when compressed at 10,000 psi, but would not when compressed at 125,000 psi. At 50 psi N_2 pressure the burning rate decreases from 0.2 ips at 10,000 psi forming pressure to 0.09 ips at 125,000 psi forming pressure. Burning rates increased with pressure in the same general fashion as for $CsNO_3$:Al.

Preliminary ionization measurements were made by means of a spectroscopic method suggested and theoretically treated by Inglis and Teller (1),[4] and experimentally checked and used by Mohler (2). The electric field produced by the ions and electrons in the vicinity of a neutral emitting hydrogen-like atom, such as cesium, broadens the closely spaced energy levels of the atom near the series limit into a continuum; the effect is proportional to the density of the ions. By determining the last discernible discrete level, one can obtain the continuum width and thus the ion density. At atmospheric pressure with both a quartz and a grating spectrograph, studies have been made of the principal and diffuse series of cesium in the combustion gases. Tentative results indicate that the ion density at atmospheric pressure for both the $CsClO_4$:Al and $CsNO_3$:Al mixtures has an upper limit roughly one half an order of magnitude below the calculated values. Further measurements of ion density by this and other methods are being continued.

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[3] Director, Kinetics and Combustion Division. Member ARS.

[4] Numbers in parentheses indicate References at end of paper.

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1 Inglis, D. R. and Teller, E., "The Number of Lines in a Series as a Function of Electron Pressure," *Astrophys. J.*, vol. 90, no. 3, Oct. 1939, pp. 429–438.

2 Mohler, F. L., "Ionic Depression of Series Limits in One-Electron Spectra," *Astrophys. J.*, vol. 90, no. 3, Oct. 1939, pp. 439–448.

Table 1 Combustion products from stoichiometric $CsNO_3$:Al[a]

Pressure, psi	Temperature, K	Cs^+, %	e, %	O^-, %
14.7	3715	2.48	2.44	0.043
121	4130	1.90	1.80	0.096
1000	4605	1.41	1.24	0.171
121 → 14.7[b]	3656	2.33	2.30	0.033
1000 → 14.7[b]	3573	2.07	2.04	0.022

[a] Other species considered include Cs, Cs_2, Al, Al_2O_2, Al_2O, AlO, AlN, N, N_2, NO, O, O_2 and Al_2O_3 (liq.). Results are expressed as mole percentages of gaseous products.

[b] After isentropic nozzle expansion from initial pressure to 14.7 psi.

Table 2 Combustion products from stoichiometric $CsClO_4$:Al[a]

Pressure, psi	Temperature, K	Cs^+, %	e, %	Cl^-, %	O^-, %
14.7	3826	3.43	0.878	2.53	0.025
121	4315	2.96	0.582	2.33	0.057
1000	4943	2.58	0.385	2.14	0.121
121 → 14.7[b]	3804	3.54	0.819	2.71	0.023
1000 → 14.7[b]	3776	3.63	0.748	2.87	0.019

[a] Other species considered include Cs, Cs_2, CsCl, Cs_2Cl^+, Al, Al_2O_2, Al_2O, AlO, AlOCl, $AlCl_3$, $AlCl_2$, AlCl, Cl, Cl_2, O, O_2 and Al_2O_3 (liq.). Results are expressed as mole percentages of gaseous products.

[b] After isentropic nozzle expansion from initial pressure to 14.7 psi.

Location of Focal Points of Ellipses

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With a strip of paper and a pencil, the focal point of any ellipse can be located to within graphical accuracy in a few seconds. This and one other method serve as effective checks on the gross errors that sometimes occur in diagrams of orbits.

OCCASIONALLY, in "blackboard talks" or published articles (1,2),[2] one notes gross errors in the location of the focal points of ellipses.[3] Through some psychological orientation toward circular symmetry, engineers and scientists alike are apt to place the focus near the center of curvature despite a distance error that may amount to 100 per cent. It is the intention here to suggest simple checks, which the editor, or any reader, can make on the spot.

Scientists sometimes excuse their carelessness in locating the focus on the grounds that it does not upset the validity of the theory they are expounding. While the pure mathematician is entitled to a measure of poetic license, it would be straining the bounds of tolerance, for example, to accept an error of 100 per cent in the perigee distance of an orbit. Engineers can fall into the same error, e.g., in a blackboard talk concerning parabolic reflector-type antennas. In this case, the slip is usually corrected before it gets into print, thanks to the shop drawings made for the design and test of the prototype.

Without going into the mathematics of ellipses, let us consider methods of locating the focus. One of the properties of the ellipse is illustrated in Fig. 1, where FB, the distance from the focus to an extremity B of the minor diameter, equals the semi-major axis PO. Location of the focal point F may be made simply, as follows.

Placing the edge of a strip of paper along the line of apsides (major diameter), mark off on the strip the perigee and apogee points P and A. Fold the paper in two, superposing P and A. Now crease to locate the center O of the ellipse. With the paper still folded, superpose the P and A points on B, holding in place by the point of a pencil. Rotate the other end O until it crosses the axis at the point F. Mark the focal point F. Thus, in a few seconds, one has checked whether the author calculated or guessed at the position of F.

A further useful check is as follows. Draw a vertical line through F intersecting the ellipse at E. The line segment FE is the semilatus rectum P, which is the *radius of curvature* of the osculating circle at the vertex P. Lay off PC on the axis equal to FE and note whether the circle about C as center is really the osculating circle at the vertex of the ellipse.

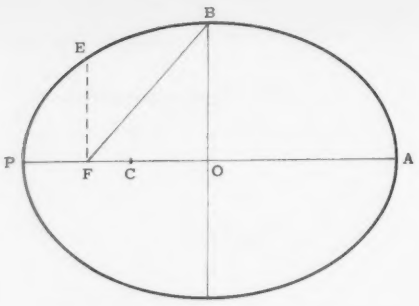

Fig. 1 Position of focus of ellipse

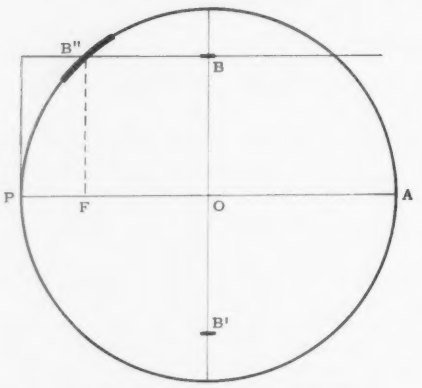

Fig. 2 Focus and auxiliary circle

A method of locating the focus F—one which is particularly good in the layout stage—is shown in Fig. 2. Assume that an ellipse is to be constructed with PA as major diameter and BB' as minor diameter. Extend the line BB'' parallel to the major axis to intersect the auxiliary circle at B''. A vertical line through B'' will then intersect the major diameter in the focus F. The proof is simple, since the opposite diagonals FB and OB'' of the rectangle $FOBB''$ are equal.

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[2] Numbers in parentheses indicate References at end of paper.

[3] The two cases mentioned in (1 and 2) are striking examples. Both appeared in the same issue of ARS JOURNAL.

References

1 Rider, L., "Ascent From Inner Circular to Outer Co-Planar Elliptic Orbits," ARS JOURNAL, vol. 30, no. 3, March 1960, p. 254, Fig. 1.

2 Pistiner, J. S., "Some Characteristics of the Planar Satellite Orbit," ARS JOURNAL, vol. 30, no. 3, March 1960, p. 275, Fig. 1.

Aerodynamic Heating of Plane Bodies of Low Thermal Diffusivity

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This note presents equations which give the transient temperature at various locations of a plane body when the adiabatic wall temperature, film coefficient and radiant heat flux are predescribed functions of time. The equations contain memory coefficients which are defined by the configuration of the body under consideration. Coefficients that are suitable for short heating times of thermally thick materials are presented for an insulated flat plate, semi-infinite composite body, composite flat plate and a semi-infinite body.

PRESENTED in (1)[2] is the development of an excellent method of calculating the temperatures of the heated surface of an insulated flat plate or a semi-infinite body when the radiation and convection boundary conditions are predescribed functions of time. The method solves the diffusion equation of transient heat transfer by employing a finite difference integration of time and an exact integration of distance. In (1), Hill obtains the exact integration in the form of a Fourier series which converges slowly for small values of

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[2] Numbers in parentheses indicate References at end of paper.

dimensionless time. The present note uses the Laplace transform method as an approach to the exact solution. This results in an infinite series containing integral error functions which converge rapidly for small times, thus facilitating numerical evaluation when the time increment is small or the material is thermally thick. These results also facilitate the investigation, under any circumstances, of the early temperature rise.

For unidirectional heat flow in an insulated flat plate with time dependent boundary conditions, Hill (1) has obtained an explicit equation for the heated wall surface temperature which is a function only of heat input, a set of memory coefficients and the heated wall temperature at the previous increment of time. The equation is

$$T_m = \frac{H_m T_{aw,m} + H_{m-1}(T_{aw,m-1} - T_{m-1}) - M_2 T_{m-1} - \ldots - M_m T_1 - R_m - R_{m-1}}{M_1 + H_m} \quad [1]$$

for a uniform initial temperature of zero, and where

$$H_m = Kh_m$$
$$R_m = KQ_{r,m}$$
$$t = m\delta \qquad m = 1, 2, 3, \ldots \quad [2]$$

and the subscript m indicates the time at which the boundary conditions T_{aw}, h, Q_r are evaluated.

The memory coefficients are defined by

$$M_m \equiv A_m - 3A_{m-1} + 3A_{m-2} - A_{m-3} \quad [3]$$

The temperatures at any plane within the interior of the solid are

$$TI_m = T_m - \Theta_1 T_m - \Theta_2 T_{m-1} - \ldots - \Theta_m T_1 \quad [4]$$

where these memory coefficients are a function of the location of the plane in question, and are defined by

$$\Theta_m \equiv \Theta_{r,m} - 2\Theta_{r,m-1} + \Theta_{r,m-2} \quad [5]$$

The terms in the memory coefficients are, for the insulated flat plate of Fig. 1

$$A_m = m^{3/2} \sum_{n=0}^{\infty} (-1)^n \left(i^3 \operatorname{erfc} \frac{n}{\sqrt{\tau m}} - i^3 \operatorname{erfc} \frac{n+1}{\sqrt{\tau m}} \right) \quad [6]$$

$$\Theta_{r,m} = m \left\{ 1 - 4 \sum_{n=0}^{\infty} (-1)^n \left[i^2 \operatorname{erfc} \frac{2n+X}{2\sqrt{\tau m}} + i^2 \operatorname{erfc} \frac{2(n+1)-X}{2\sqrt{\tau m}} \right] \right\} \quad [7]$$

and

$$K = \frac{a}{k} \frac{\sqrt{\tau}}{16} \quad [8]$$

The terms of the memory coefficients for the composite flat plate of Fig. 2 are as follows:

$$A_m = m^{3/2} \sum_{n=0}^{\infty} (-1)^n \left[i^3 \operatorname{erfc} \frac{n(\beta L+1)}{\sqrt{\tau m}} - i^3 \operatorname{erfc} \frac{(n+1)(\beta L+1)}{\sqrt{\tau m}} \right] \quad [9]$$

For $0 < X \leq 1$

$$\Theta_{r,m} = m \left\{ 1 - 4 \sum_{n=0}^{\infty} (-1)^n \left[i^2 \operatorname{erfc} \frac{2n(\beta L+1)+X}{2\sqrt{\tau m}} + i^2 \operatorname{erfc} \frac{2(n+1)(\beta L+1)-X}{2\sqrt{\tau m}} \right] \right\} \quad [10]$$

For $1 < X \leq L+1$

$$\Theta_{r,m} = m \left\{ 1 - 4 \sum_{n=0}^{\infty} (-1)^n \left[i^2 \operatorname{erfc} \frac{(2n+1)(\beta L+1) - \beta(1+L-X)}{2\sqrt{\tau m}} + i^2 \operatorname{erfc} \frac{(2n+1)(\beta L+1) + \beta(1+L-X)}{2\sqrt{\tau m}} \right] \right\} \quad [11]$$

and $K = (a/k_1)(\sqrt{\tau}/16)$, which is Equation [8]. A condition for the foregoing equations is that

$$\gamma = \frac{k_2}{k_1} \sqrt{\frac{\alpha_1}{\alpha_2}} \approx 1$$

The memory coefficients for a semi-infinite composite body (Fig. 3) are

$$A_m = m^{3/2} \sum_{n=0}^{\infty} \frac{1}{A^n} \left(i^3 \operatorname{erfc} \frac{n}{\sqrt{\tau m}} + \frac{1}{A} i^3 \operatorname{erfc} \frac{n+1}{\sqrt{\tau m}} \right) \quad [12]$$

For $0 < X \leq 1$

$$\Theta_{r,m} = m \left\{ 1 - 4 \sum_{n=0}^{\infty} \frac{1}{A^n} \left[i^2 \operatorname{erfc} \frac{2n+X}{2\sqrt{\tau m}} - \frac{1}{A} i^2 \operatorname{erfc} \frac{2(n+1)-X}{2\sqrt{\tau m}} \right] \right\} \quad [13]$$

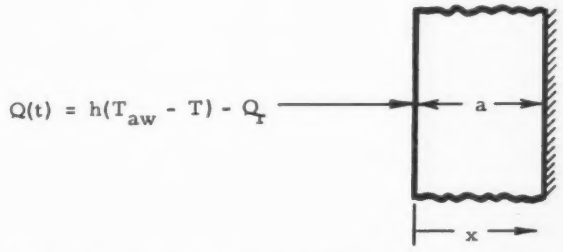

Fig. 1 Coordinates for the insulated flat plate with a time dependent boundary condition

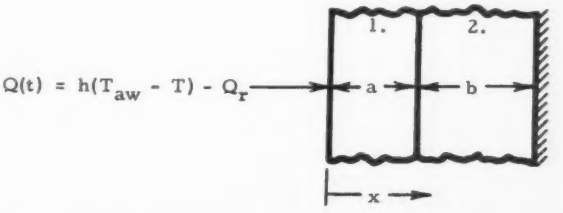

Fig. 2 Coordinates for the composite insulated flat plate with time dependent boundary conditions

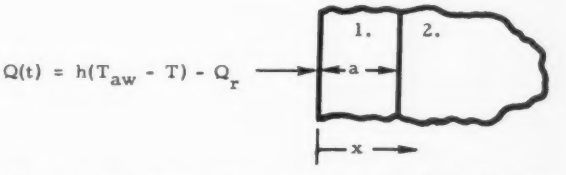

Fig. 3 Coordinates for the composite semi-infinite solid with time dependent boundary conditions

For $1 < X < \infty$

$$\Theta_{r,m} = m\left[1 - \frac{8}{1+\gamma}\sum_{n=0}^{\infty}\frac{1}{A^n}i^2 \times \operatorname{erfc}\frac{2(n+1)+\beta(X-1)}{2\sqrt{\tau m}}\right] \quad [14]$$

where $K = (a/k_1)(\sqrt{\tau}/16)$ (Eq. [8]), and with the restriction that $\gamma \neq 1$.

At this point an interesting observation can be made regarding the similarity between the memory coefficients of the bodies in Figs. 1–3. Assume that the first layer in the composite bodies is to be insulated. This can be mathematically achieved by letting the thickness of plate 2 in Fig. 2 be equal to zero and the thermal conductivity of the semi-infinite body 2 in Fig. 3 be equal to zero. Thus, the insulated flat plate requirements are satisfied, and the memory coefficients become identical to the memory coefficients of the insulated flat plate of Fig. 1.

The memory coefficients for the semi-infinite solid of Fig. 4 are

$$A_m = (1/6\sqrt{\pi})m^{3/2} \quad [15]$$

$$\Theta_{r,m} = m\left[1 - 4i^2\operatorname{erfc}\frac{1}{2\sqrt{(\alpha\delta/x^2)m}}\right] \quad [16]$$

where

$$K = \frac{x}{16k}\sqrt{\frac{\alpha\delta}{x^2}}$$

The integral error functions facilitate the investigations of the early temperature rise in the bodies because of the rapid convergence of the infinite series for small values of dimensionless time. This is in direct contrast with the Fourier series solution of (1) which converges slowly for small values of dimensionless time.

From (2)

$$i^3\operatorname{erfc}\xi \cong 0 \qquad \xi \geq 3 \quad [18]$$

$$i^3\operatorname{erfc}0 = 1/6\sqrt{\pi} \quad [19]$$

Therefore, Equation [6] for a flat plate reduces to

$$A_m = \frac{1}{6\sqrt{\pi}}m^{3/2} \qquad (\tau m \leq \tfrac{1}{9}) \quad [20]$$

It now can be seen that the memory coefficient terms for a flat plate are identical to the terms for a semi-infinite solid, realizing that $x = a$ in the K factor. This important result indicates, then, that the temperature of the heated surface of the bodies in Figs. 1–3 behaves exactly like that of a semi-infinite solid for the condition that

$$\tau m \leq \tfrac{1}{9} \quad [21]$$

indicating to which time the closed form solution of the semi-infinite solid is applicable for solving for the surface temperatures of flat plates.

Another interesting observation can also be made for small values of τ. It is possible to show that the relationship between the memory coefficients of this note and those of (1) is

$$M_m = (1/\pi^2\sqrt{\tau})M'_m \quad [22]$$

where the prime refers to values in the tables of (1).

Also, on inspecting (1) and comparing the semi-infinite memory coefficients with those of the present note (Eq. [15]) it is found that

$$M'_{\infty,m} = 6\sqrt{\pi}\,M_{\infty,m} \quad [23]$$

Since

$$M_{\infty,m} = M_m \qquad (rm \leq \tfrac{1}{9}) \quad [24]$$

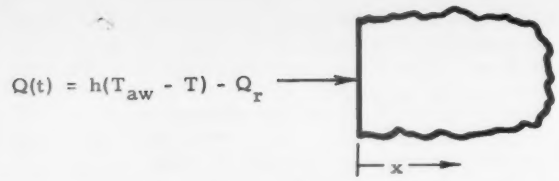

Fig. 4 Coordinates for the semi-infinite solid with time dependent boundary conditions

where ∞ refers to the memory coefficients for the semi-infinite body, we now have

$$M'_{\infty,m} = (6\pi^{-3/2}/\sqrt{\tau}M'_m) \quad [25]$$

which is the relationship between the values of M'_m in Table I of (1) and the values of $M'_{\infty,m}$ on page 15 of (1) for $\tau m \leq \frac{1}{9}$.

Appendix

The complementary error function is defined by

$$\operatorname{erfc} z = (2/\sqrt{\pi})\int_z^{\infty} e^{-\xi^2}\,d\xi$$

The repeated integrals of the error function are

$$i^n\operatorname{erfc} z = \int_z^{\infty} i^{n-1}\operatorname{erfc}\xi\,d\xi \qquad n = 1, 2, 3, \ldots$$

with

$$i^0\operatorname{erfc} z = \operatorname{erfc} z$$

Integrating by parts gives

$$i\operatorname{erfc} z = (1/\sqrt{\pi})e^{-z^2} - z\operatorname{erfc} z$$

from which arises the general recurrence formula

$$2ni^n\operatorname{erfc} z = i^{n-2}\operatorname{erfc} z - 2zi^{n-1}\operatorname{erfc} z$$

A comprehensive table of the repeated integrals of the error function is contained in (2).

Nomenclature

T	=	temperature at the heated surface
T_{aw}	=	adiabatic wall temperature
h	=	convective heat transfer coefficient
Q_r	=	heat lost by radiation per unit area
$Q(t)$	=	time dependent heat input
t	=	time
δ	=	time increment
TI	=	temperature in the interior of the solid
a	=	thickness defined in Figs. 1–3
b	=	thickness defined in Fig. 2
x	=	distance into the solid
α	=	thermal diffusivity
k	=	thermal conductivity
τ	=	$\alpha_1\delta/a^2$
β	=	$\sqrt{(\alpha_1/\alpha_2)}$
γ	=	$(k_2/k_1)\beta$
A	=	$(\gamma+1)/(\gamma-1)$
X	=	x/a
L	=	b/a

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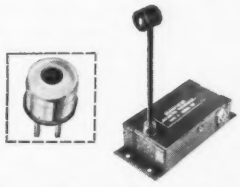

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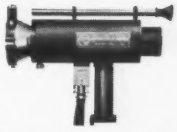

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New Patents

George F. McLaughlin, *Associate Editor*

Rocket propellant igniter (2,937,493). B. R. Adelman (ARS member), Waco Texas, assignor to Phillips Petroleum Co.

Ignitor material in a container directing part of its ignition products to the internal and external surfaces of the propellant grain.

Multiple thermal jet propulsion system (2,937,494). H. J. Johnson (ARS member), Grosse Pointe, Mich.

Series type consisting of a forward unit and a trailing unit. Velocity of the gas flowing from the outlet of the forward unit is maintained constant while the volume of gases decreases and their density increases.

Jet propelling unit (2,937,496). G. Caillete, Neuilly-sur-Seine, France.

Two fore-and-aft extending turbojet power units aligned with their inlets facing each other. The front unit receives air from the atmosphere at its rearward end.

Explosion test sampling chamber (2,937,530). J. C. Haley, Dayton, Ohio, assignor to the U. S. Air Force.

Closed chamber enclosing a sampling chamber of smaller volume in which a gaseous mixture can be ignited to determine if it is instantaneously explosive.

Gyroscope caging mechanism (2,937,531). C. E. Gallagher, Stockbridge, Mass.

For control of pilotless aircraft by remote signals. The caging means rides in an aperture in a drum which is synchronized with the caging motor.

Erector-launching apparatus (2,937,574). H. E. Senft, Schenectady, N. Y., assignor to Alco Products, Inc.

Rotatable shaft on a stand supporting launching rails which may be raised at an angle up to 90 deg.

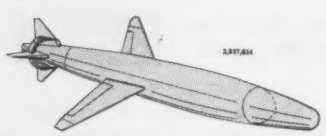
Rocket boosters (2,937,595). W. Margulis and M. S. Silvey, Schenectady, N. Y., assignors to Alco Products.

Fusible nut on a bolt released by thermal action of propellant to jettison the rocket container from the rocket.

Missile nose structure (2,937,597). H. Winn and S. Levy (ARS member), Bala Cynwyd, Pa., assignors to General Electric Co. (ARS corporate member).

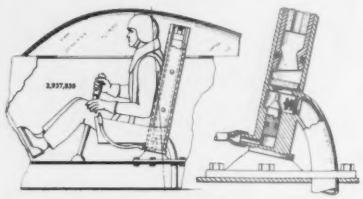
Nosecone structure supported so that upon expansion of the skin by heat, the support prevents buckling, flutter and destruction of the skin.

Bimedium rocket-torpedo missile (2,937,824). S. G. Krumbholz (ARS member) and G. M. McRoberts III, Sierra Madre, Calif., assignors to Aerojet-General Corp. (ARS corporate member).



Missile adapted to travel through air, driven by a rocket motor, and through water, driven by a water pump. Fuel and oxidizer to the motor are cut off when the missile enters the water.

Ejecting device for aircraft (2,937,830). N. D. Fulton (ARS member), Barnardsville, N. J., assignor to Thiokol Chemical Corp. (ARS corporate member).



Rocket catapult tube for ejecting a pilot from an aircraft in an emergency. Exhaust gases of the rocket act against the reactive surface of the tube to provide high initial acceleration.

Pellet accelerometer (2,938,087). J. M. Meek, Silver Spring, Md., assignor to the U. S. Army.

Device responsive to acceleration and adapted to initiate arming of a missile some predetermined time after the launching.

Rocket propellant (2,938,330). J. J. Kolfenbach (ARS member) and R. F. Finn, Iselin, N. J., assignors to Esso Research and Engineering Co. (ARS corporate member).

Nitric acid oxidizer and a hydrocarbon polymer introduced into the combustion chamber of a rocket engine.

Combustion chamber liner (2,938,333). J. M. Wetzler, Indianapolis, Ind., assignor to General Motors Corp. (ARS corporate member).

Inverted domes edge-interconnected with each other in quiltlike fashion to provide an annular cellular chamber about the combustion chamber.

Supersonic inlet (2,938,334). G. H. McLafferty, Manchester, Conn., assignor to United Aircraft Corp. (ARS corporate member).

Air intake for high speed aircraft. Scoop protruding from the wall adjacent a restricted throat for removing boundary layer flow.

Combustion flask for liquid oxidants (2,938,343). R. H. Canfield, Newport, R. I., assignor to the U. S. Navy.

Centrifugal separating means arranged to direct lighter components of comburent materials inward into the combustion chamber, and heavier components longitudinally in the form of a film.

Jet propelled torpedo (2,938,481). L. R. Maxwell and R. K. Reber, Washington, D. C.

Valves admit water-reactive chemical into the explosive chamber causing intermittent explosions which close inwardly opening ports and expel the water through an exhaust to propel the device.

Solid propellant starter for liquid fuel monopropellant (2,938,482). F. A. Newburn, R. D. Cannova (ARS members) and B. I. Friedman, Sunnyvale, Calif., assignors to the U. S. Navy.

Torpedo having an external combustion prime mover operated by decomposition gases of a liquid monopropellant injected into the decomposition chamber and into proximity with burning solid propellant.

Analysis of radar meteorological statistical data (2,939,129). D. Atlas, L. Eisenberg, W. Hitchfeld and J. S. Marshall, Montreal, Canada, assignors to the U. S. Air Force.

Train of vertical pulses transmitted from a fixed ground station to a storm cloud, and receiving the echo reflected by the moisture content of the cloud.

Airplane design (188,079). L. B. Rellis, Chicago, Ill.

Delta wing aircraft with jet engine above the fuselage fin and four jet engines mounted on a separate horizontal tail surface.

Pressurized suit (2,939,148). T. H. Hart, Norristown, Pa.

Fluid-tight universally articulated joints of unrestricted internal passage, and flexible baglike tubular members.

Solid fuel rocket-type motor (2,939,275). A. C. Loedding (ARS member), Princeton, N. J., assignor to Unexcelled Chemical Corp.

Cylindrical container with charge of solid propellant, the charge comprising adjacent elongated rods of hexagonal cross section.

Engine diffuser spike positioning system (2,939,277). J. C. Wise, Woodland Hills, Calif., assignor to The Marquardt Corp.

Movable means at the inlet for varying the area and for obtaining a single static pressure at the diffuser lip.

Method for starting rocket motors (2,939,278). H. M. Fox (ARS member), Bartlesville, Okla., assignor to Phillips Petroleum Co.

Pressure sensitive means connected to the combustion chamber and to the fuel supply to simultaneously and gradually start and stop the hypergolic fuel supply and the nonhypergolic fuel supply when a predetermined pressure is attained in the combustion zone.

Jet noise suppressor (2,939,279). A. L. Highberg, West Hartford, Conn.. assignor to United Aircraft Corp. (ARS corporate member).

Ducts mounted for rotation on a circular shell, each culminating in a slot which cooperates with similar slots in all other ducts to form an outlet for the gases.

Rocket grain (2,939,396). B. R. Adelman (ARS member), Waco, Texas, assignor to Phillips Petroleum Co.

Rectangular propellant material, the grain having at least one centrally disposed longitudinally extending perforation in a rodlike support member.

Rotating jet aircraft (2,939,648). H. Fleissner, Dansing, Germany, assignor of one half P. O. Tobeler (Trans-Oceanic), Los Angeles, Calif.

Wing comprising a double convex-

EDITOR'S NOTE: Patents listed above were selected from the Official Gazette of the U. S. Patent Office. Printed copies of patents may be obtained from the Commissioner of Patents, Washington 25, D. C., at a cost of 25 cents each; design patents, 10 cents.

shaped disk having a radially outward directed taper from its center. Bucket vanes receive exhaust from rockets.

Device for control of lifting and directional elements (2,939,654). H. Coanda, Paris, France, assignor to Sebac Nouvelle S.A. Corp.

Jet tubes angularly spaced on the circumference of a circle arranged for simultaneous tilting within a predetermined extent.

Aircraft landing mechanism (2,939,657). W. B. Wescott Jr., Cleveland Heights, Ohio, assignor to Cleveland Pneumatic Industries.

Probe on the nose of an aircraft for engagement with a supporting frame and landing platform. Attached to the side of an aircraft carrier, the device may be swung to the deck after engagement.

Control of pressure in rocket motors (2,940,251). S. S. Prentiss, Bartlesville, Okla., assignor to Phillips Petroleum Co.

Elliptical exhaust nozzle in the end wall of a rocket motor shell adapted to be changed into a round nozzle by outward deformation of the wall.

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M. H. Smith, *Associate Editor*
The James Forrestal Research Center, Princeton University

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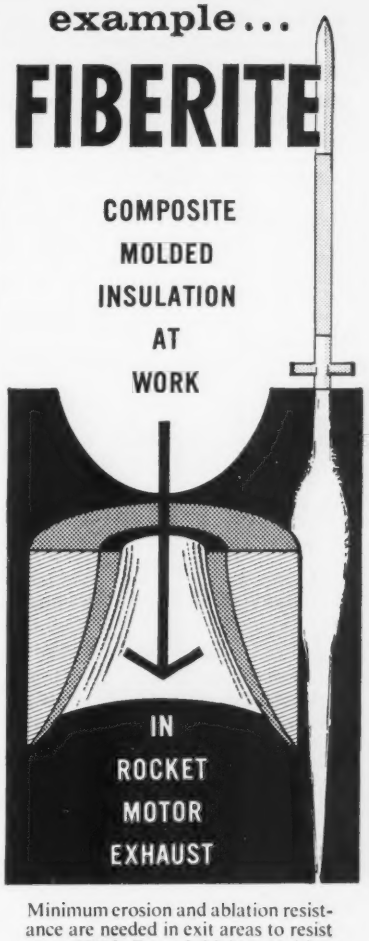

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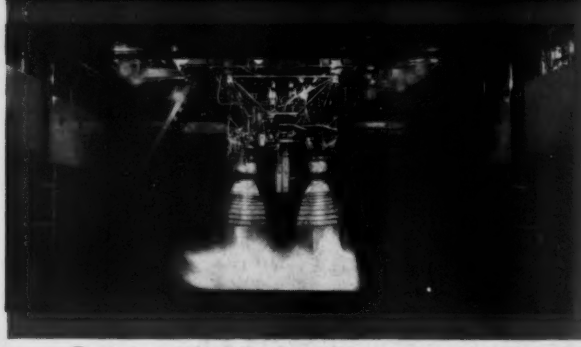

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